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Journal of Functional Analysis 218 (2005) 54–109

JOURNAL OF Functional Analysis

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Hardy Sobolev spaces on strongly Lipschitz domains of $\mathbb{R}^n$

Pascal Auscher[a,*], Emmanuel Russ[b,c], Philippe Tchamitchian[b,c]

[a] CNRS UMR 8628, Université de Paris-Sud, 91405 Orsay Cedex, France

[b] Faculté des Sciences et Techniques de Saint-Jérôme, Avenue Escadrille Normandie-Niémen, 13397 Marseille Cedex 20, France

[c] LATP, CNRS, UMR 6632, France

Received 7 July 2003; accepted 7 June 2004

Communicated by M. Christ

Available online 5 October 2004

Abstract

Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. The Hardy spaces $H^1_r(\Omega)$ and $H^1_z(\Omega)$ have been introduced by Miyachi (Studia Math. 95(3) (1990) 205), Jonsson et al. (Studia Math. 80(2) (1984) 141) and Chang et al. (J. Funct. Anal. 114 (1993) 286). We first investigate spaces of functions in $L^1(\Omega)$ whose gradients belong to $H^1_r(\Omega)$ or $H^1_z(\Omega)$, which we call Hardy–Sobolev spaces, following Strichartz (Coll. Math. 60–61(1) (1990) 129).

Secondly, if $L = -\text{div}\, A\nabla$ is a uniformly elliptic second-order divergence operator on Ω with measurable complex coefficients subject to the Dirichlet or the Neumann boundary condition, we compare the norms of $L^{1/2}f$ and ∇f in suitable Hardy spaces on Ω, depending on the boundary condition, under the assumption that the heat kernel of L satisfies suitable estimates.

1. Introduction

Throughout this paper, Ω denotes a strongly Lipschitz domain of $\mathbb{R}^n$ ($n \geqslant 2$), i.e. an open connected set in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is a finite union of parts of rotated graphs of Lipschitz maps, at most one of these parts possibly unbounded. We denote by $d\sigma$ the surface measure on $\partial\Omega$.

* Corresponding author.

E-mail address: pascal.auscher@math.u-psud.fr (P. Auscher).

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.06.005

Hardy spaces on Ω have been studied in Refs. [22,10,9,3]. Here, we just give basic definitions. We first recall what $H^1(\mathbb{R}^n)$ is. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. For all $t > 0$, define $\phi_t(x) = t^{-n}\phi(x/t)$. A locally integrable function f on $\mathbb{R}^n$ is said to be in $H^1(\mathbb{R}^n)$ if the vertical maximal function

$$\mathcal{M}f(x) = \sup_{t>0} |\phi_t * f(x)|$$

belongs to $L^1(\mathbb{R}^n)$. If it is the case, define

$$\|f\|_{H^1(\mathbb{R}^n)} = \|\mathcal{M}f\|_1.$$

Note that $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and that a function in $H^1(\mathbb{R}^n)$ always has zero integral.

One may consider two Hardy spaces on Ω built upon $H^1(\mathbb{R}^n)$. Their definitions mimic the ones of $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$. A function f on Ω is said to be in $H_r^1(\Omega)$ if it is the restriction to Ω of a function $F \in H^1(\mathbb{R}^n)$. If $f \in H_r^1(\Omega)$, define $\|f\|_{H_r^1(\Omega)}$ by

$$\|f\|_{H_r^1(\Omega)} = \inf \|F\|_{H^1(\mathbb{R}^n)},$$

the infimum being taken over all the functions $F \in H^1(\mathbb{R}^n)$ such that $F|_\Omega = f$. Equipped with that norm, $H_r^1(\Omega)$ is a Banach space. In other words,

$$H_r^1(\Omega) = H^1(\mathbb{R}^n)/\left\{f \in H^1(\mathbb{R}^n);\;\; f = 0 \text{ on } \Omega\right\}.$$

Define also

$$H_z^1(\Omega) = \left\{f \in H^1(\mathbb{R}^n);\;\; \operatorname{Supp} f \subset \overline{\Omega}\right\}.$$

Endowed with the $H^1(\mathbb{R}^n)$ norm, it is a Banach space, which is strictly contained in $H_r^1(\Omega)$. Observe, in particular, that, for all $f \in H_z^1(\Omega)$, one has $\displaystyle\int_\Omega f(x)\,dx = 0$, whereas this may not happen for $f \in H_r^1(\Omega)$. Let us also mention that $H_z^1(\Omega)$ is equal to the atomic Hardy space of Coifman–Weiss defined in [12], since Ω is a space of homogeneous type (see [3] for this fact).

If $\mathbf{F} = (F_1, \ldots, F_n)$ is a $\mathbb{C}^n$-valued function defined on Ω, say that $\mathbf{F} \in H_r^1(\Omega)$ (resp. $H_z^1(\Omega)$) if, for all $1 \leqslant i \leqslant n$, $F_i \in H_r^1(\Omega)$ (resp. for all $1 \leqslant i \leqslant n$, $F_i \in H_z^1(\Omega)$). Define

$$\|\mathbf{F}\|_{H_r^1(\Omega)} = \sum_{i=1}^n \|F_i\|_{H_r^1(\Omega)} \quad \text{(resp.} \quad \|\mathbf{F}\|_{H_z^1(\Omega)} = \sum_{i=1}^n \|F_i\|_{H_z^1(\Omega)}\text{)}.$$

In this paper, we investigate spaces of functions $f \in L^1(\Omega)$ such that $\nabla f \in H^1_r(\Omega)$ or $\nabla f \in H^1_z(\Omega)$, and their corresponding homogeneous versions. We call them Hardy–Sobolev spaces. When $\Omega = \mathbb{R}^n$, this space is well understood (see [27]). We thus set

$$H^{1,1}_r(\Omega) = \left\{ f \in L^1(\Omega);\ \nabla f \in H^1_r(\Omega) \right\},$$

equipped with the norm

$$\|f\|_{H^{1,1}_r(\Omega)} = \|f\|_{L^1(\Omega)} + \|\nabla f\|_{H^1_r(\Omega)} = \|f\|_{L^1(\Omega)} + \sum_{1 \leqslant i \leqslant n} \left\| \partial_i f \right\|_{H^1_r(\Omega)}$$

and

$$H^{1,1}_z(\Omega) = \left\{ f \in L^1(\Omega);\ \nabla f \in H^1_z(\Omega) \right\},$$

equipped with the corresponding norm.

Hardy–Sobolev spaces are contained in $W^{1,1}(\Omega)$, on which the trace operator to $\partial\Omega$ is well-defined onto $L^1(\partial\Omega)$ (see [23, Théorème 4.2, p. 84]). As in the case of classical Sobolev spaces, we define two more Hardy–Sobolev spaces on Ω. Set

$$H^{1,1}_{r,0}(\Omega) = \left\{ f \in L^1(\Omega);\ \mathrm{Tr}\, f = 0 \text{ on } \partial\Omega \text{ and } \nabla f \in H^1_r(\Omega) \right\}$$

and

$$H^{1,1}_{z,0}(\Omega) = \left\{ f \in L^1(\Omega);\ \mathrm{Tr}\, f = 0 \text{ on } \partial\Omega \text{ and } \nabla f \in H^1_z(\Omega) \right\}$$

equipped with the corresponding norms. These four Hardy–Sobolev spaces are clearly Banach spaces.

We also consider "homogeneous" versions of these Hardy–Sobolev spaces. Say that a measurable function f on Ω belongs to $L^1_c(\Omega)$ if, for all compact $K \subset \mathbb{R}^n$, $\displaystyle\int_{K \cap \Omega} |f(x)|\, dx < +\infty$. Define

$$\dot{H}^{1,1}_r(\Omega) = \left\{ f \in L^1_c(\Omega);\ \nabla f \in H^1_r(\Omega) \right\},$$

equipped with the "homogeneous" semi-norm

$$\|f\|_{\dot{H}^{1,1}_r(\Omega)} = \|\nabla f\|_{H^1_r(\Omega)}$$

and

$$\dot{H}_z^{1,1}(\Omega) = \left\{ f \in L_c^1(\Omega);\ \ \nabla f \in H_z^1(\Omega) \right\},$$

equipped with the “homogeneous” semi-norm

$$\|f\|_{\dot{H}_z^{1,1}(\Omega)} = \|\nabla f\|_{H_z^1(\Omega)} .$$

As we shall see several times in the paper, only $H_r^{1,1}(\Omega)$ and $H_{z,0}^{1,1}(\Omega)$ are “natural”. In particular, they are the only spaces which can be characterized in terms of “adapted” maximal functions and of extension theorems. They are also the convenient spaces for the boundedness properties at $p = 1$ of square root of elliptic operators.

This paper is divided in two parts. In the first part, we make a systematic study of these Hardy–Sobolev spaces on Ω. To begin with, we explain basic ideas and give our first results in the case when Ω is the upper half space (Section 2). Then, we characterize the Hardy–Sobolev norms in terms of maximal functions, which are adapted to the analysis of a gradient vector field (Section 3). Several applications are given next. We obtain endpoint versions of the div-curl lemma, showing new improved regularity results in these spaces (Section 4). Then, in Section 5, we investigate various properties of Hardy–Sobolev spaces in the spirit of classical Sobolev spaces: change of variables, extension and restriction theorems, density, duality, interpolation with classical Sobolev spaces.

In the second part (Section 6), we apply our results to give endpoint estimates at $p = 1$ for the square root of elliptic second-order divergence operators on Ω with measurable complex coefficients subject to the Dirichlet or the Neumann boundary condition. By interpolation arguments, we also recover the L^p theory already obtained in [7].

Notation 1. *If $A(f)$ and $B(f)$ are two non-negative quantities depending on a function f belonging to some space E, the notation $A(f) \sim B(f)$ means that there exists $C > 0$ such that, for all $f \in E$,*

$$C^{-1} A(f) \leqslant B(f) \leqslant C A(f).$$

2. The example of the upper half space

As a tutorial section, we investigate some properties of Hardy–Sobolev spaces on

$$\mathbb{R}_+^n = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ \ x_n > 0 \right\}.$$

2.1. A reflection principle

For all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, set

$$Sx = (x_1, \ldots, x_{n-1}, -x_n).$$

If f is any function defined on $\mathbb{R}^n_+$, its even extension f_e is defined on $\mathbb{R}^n$ by

$$f_e(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^n_+, \\ f(Sx) & \text{if } x \in \mathbb{R}^n_-, \end{cases}$$

whereas its odd extension f_o is given by

$$f_o(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^n_+, \\ -f(Sx) & \text{if } x \in \mathbb{R}^n_-, \end{cases}$$

where

$$\mathbb{R}^n_- = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n;\ \ x_n < 0 \right\}.$$

The reflection principle for Hardy–Sobolev spaces is as follows:

Proposition 2. *Let* $f \in L^1_c(\mathbb{R}^n_+)$.

(a) *One has* $\nabla f \in H^1_r(\mathbb{R}^n_+)$ *iff* $\nabla f_e \in H^1(\mathbb{R}^n)$. *Moreover,* $\|\nabla f\|_{H^1_r(\mathbb{R}^n_+)} \sim \|\nabla f_e\|_{H^1(\mathbb{R}^n)}$.

(b) *If* $\mathrm{Tr}\, f = 0$ *on* $\partial\mathbb{R}^n_+$, *then* $\nabla f \in H^1_z(\mathbb{R}^n_+)$ *iff* $\nabla f_o \in H^1(\mathbb{R}^n)$. *Moreover,* $\|\nabla f\|_{H^1_z(\mathbb{R}^n_+)} \sim \|\nabla f_o\|_{H^1(\mathbb{R}^n)}$.

Before giving the proof, we point out that, if g is a locally integrable function defined on an open subset U of $\mathbb{R}^n$, for all $1 \leqslant i \leqslant n$, $\partial_i g$ denotes its derivative with respect to the ith coordinate in $\mathcal{D}'(U)$. For instance, in Proposition 2, ∇f is defined in $\mathcal{D}'(\mathbb{R}^n_+)$, whereas ∇f_e and ∇f_o are defined in $\mathcal{D}'(\mathbb{R}^n)$.

The proof relies on an improved regularity result which will be shown later (Corollary 23): if $\nabla f \in H^1_r(\mathbb{R}^n_+)$, then actually, for all $1 \leqslant i \leqslant n-1$, $\partial_i f \in H^1_z(\mathbb{R}^n_+)$.

We also make use of the scalar reflection principle for Hardy spaces on $\mathbb{R}^n_+$ (see Corollaries 1.6 and 1.8 in [10]): a complex-valued function f defined on $\mathbb{R}^n_+$ belongs to $H^1_r(\mathbb{R}^n_+)$ (resp. $H^1_z(\mathbb{R}^n_+)$) iff $f_o \in H^1(\mathbb{R}^n)$ (resp. $f_e \in H^1(\mathbb{R}^n)$), and one has

$$\|f\|_{H^1_r(\mathbb{R}^n_+)} \sim \|f_o\|_{H^1(\mathbb{R}^n)}, \quad \|f\|_{H^1_z(\mathbb{R}^n_+)} \sim \|f_e\|_{H^1(\mathbb{R}^n)}.$$

Here is the proof of Proposition 2. For all $1 \leqslant i \leqslant n-1$, $(\partial_i f_e) = (\partial_i f)_e$ and $(\partial_n f_e) = (\partial_n f)_o$. One has $\nabla f \in H^1_r(\mathbb{R}^n_+)$ if and only if $\partial_i f \in H^1_z(\mathbb{R}^n_+)$ for all $1 \leqslant i \leqslant n-1$ and $\partial_n f \in H^1(\mathbb{R}^n_+)$, and this, thanks to the scalar reflection principle, is equivalent to $(\partial_i f)_e \in H^1(\mathbb{R}^n)$ for all $1 \leqslant i \leqslant n-1$ and $(\partial_n f)_o \in H^1(\mathbb{R}^n)$, therefore equivalent to $\nabla f_e \in H^1(\mathbb{R}^n)$. This proves (a).

To prove (b), observe that $(\partial_i f_o) = (\partial_i f)_o$ for all $1 \leqslant i \leqslant n-1$ and that, when f has zero trace on $\partial\mathbb{R}^n_+$, $(\partial_n f_o) = (\partial_n f)_e$ in $\mathcal{D}'(\mathbb{R}^n)$, and conclude in the same way.

Proposition 2 already indicates that $H^{1,1}_r$ and $H^{1,1}_{z,0}$ are the natural candidates for extension theorems.

2.2. Comparisons between Hardy–Sobolev spaces on $\mathbb{R}^n_+$

It is obvious that $H^{1,1}_{z,0}(\mathbb{R}^n_+) \subset H^{1,1}_z(\mathbb{R}^n_+) \subset H^{1,1}_r(\mathbb{R}^n_+)$ and that $H^{1,1}_{z,0}(\mathbb{R}^n_+) \subset H^{1,1}_{r,0}(\mathbb{R}^n_+) \subset H^{1,1}_r(\mathbb{R}^n_+)$ and the same inclusions hold in the case of homogeneous spaces. Let us show that the previous inclusions are all strict.

Let g and h be nonzero functions in $\mathcal{D}(\mathbb{R})$ such that g is odd and $h(0) = 1$. Define, for all $(x, y) \in \mathbb{R}^2_+$, $f(x, y) = g(x)h(y)$. It is easily checked that $f \in H^{1,1}_z(\mathbb{R}^2_+)\backslash H^{1,1}_{z,0}(\mathbb{R}^2_+)$. Indeed, up to multiplicative constants, $\partial_x f$ and $\partial_y f$ are $H^1(\mathbb{R}^2)$ atoms supported in cubes included in $\overline{\mathbb{R}^2_+}$ (see [19,10]).

Consider now a function $g \in \mathcal{D}(\mathbb{R})$ with $\int_{\mathbb{R}} g(x)\,dx = 1$, and h as before. If $f(x, y) = g(x)h(y)$ for all $(x, y) \in \mathbb{R}^2_+$, $\partial_x f, \partial_y f \in H^1_r(\mathbb{R}^2_+)$ (since, up to multiplicative constants, $\partial_x f$, extended by 0 outside $\mathbb{R}^2_+$, is an $H^1(\mathbb{R}^2)$ atom and $(\partial_y f)_o$ is an $H^1(\mathbb{R}^2)$ atom), but $\partial_y f \notin H^1_z(\mathbb{R}^2_+)$ since its integral over $\mathbb{R}^2_+$ is nonzero. Thus, $f \in H^{1,1}_r(\mathbb{R}^2_+)\backslash H^{1,1}_z(\mathbb{R}^2_+)$.

The fact that the inclusion $H^{1,1}_{z,0}(\mathbb{R}^n_+) \subset H^{1,1}_{r,0}(\mathbb{R}^n_+)$ is strict is deeper:

Proposition 3. *There exists a function $f \in H^{1,1}_{r,0}(\mathbb{R}^n_+)$ such that $\nabla f \notin H^1_z(\mathbb{R}^n_+)$.*

Fix $\alpha \in]1, 2[$. Define a function $h \in C^\infty(\mathbb{R}^*)$, odd, supported in $[-1, 1]$ such that

$$h(y) = \frac{1}{y\,|\ln\, y|^\alpha} \quad \text{if } 0 < y \leqslant \frac{1}{2}$$

and

$$\int_0^1 h(y)\,dy = 0.$$

Set, for all $y \in \mathbb{R}$,

$$g(y) = \int_0^y h(t)\,dt,$$

so that g is supported in $[-1, 1]$, is continuous on $\mathbb{R}$ and satisfies $g(0) = 0$. Fix a function $\phi \in \mathcal{D}(]-1, 1[)$ with $\int \phi(t)\,dt = 1$ and let $f(x, y) = \phi(x)g(y)$ for all $(x, y) \in \mathbb{R}^2$. Then, we claim that $f|_{\mathbb{R}^2_+} \in H^{1,1}_{r,0}(\mathbb{R}^2_+)$, but $f|_{\mathbb{R}^2_+} \notin H^{1,1}_z(\mathbb{R}^2_+)$.

Indeed, $f \in L^1(\mathbb{R}^2)$ and $f = 0$ on $\partial\mathbb{R}^2_+$, so that $\nabla\left(f|_{\mathbb{R}^2_+}\right)$ is the restriction of ∇f to $\mathbb{R}^2_+$. Thus, it is enough to show that $\nabla f \in H^1(\mathbb{R}^2)$. Up to a multiplicative constant, $\partial_x f$ is an atom in $H^1(\mathbb{R}^2)$ (see [19]). Next, $\partial_y f = \phi(x)h(y)$, and we check that this function also belongs to $H^1(\mathbb{R}^2)$. Indeed, for all $j \geqslant 0$, set

$$a_j(y) = h(y)\mathbb{1}_{]-2^j,-2^{-j-1}]\bigcup[2^{-j-1},2^{-j}[}(y)$$

and, for all integer $k \in \{-2^{j-1}, \ldots, 2^{j-1}-1\}$, set

$$\phi_{j,k}(x) = \phi(x)\mathbb{1}_{[k2^{-j+1},(k+1)2^{-j+1}[}(x),$$

so that

$$\partial_y f = \sum_{j \geqslant 0} \sum_{-2^{j-1} \leqslant k < 2^{j-1}} \phi_{j,k} \otimes a_j.$$

First, $\phi_{j,k} \otimes a_j$ is supported in $Q = [k2^{-j+1}, (k+1)2^{-j+1}] \times [-2^{-j}, 2^{-j}]$. One has the estimate

$$\left\|\phi_{j,k} \otimes a_j\right\|_\infty \leqslant C\,\frac{2^j}{j^\alpha} = \frac{C'}{2^j j^\alpha}\,4^{j-1} = \frac{C'}{2^j j^\alpha}\,\frac{1}{|Q|}.$$

Moreover, $\displaystyle\int \phi_{j,k} \otimes a_j = 0$ since a_j is odd. Thus, up to a multiplicative constant, $\phi_{j,k} \otimes a_j$ is an atom of $H^1(\mathbb{R}^2)$, and

$$\left\|\phi_{j,k} \otimes a_j\right\|_{H^1(\mathbb{R}^2)} \leqslant \frac{C'}{2^j j^\alpha}.$$

As a consequence,

$$\left\|\partial_y f\right\|_{H^1(\mathbb{R}^2)} \leqslant C' \sum_{j \geqslant 0} \sum_{-2^{j-1} \leqslant k < 2^{j-1}} \frac{1}{2^j j^\alpha} < +\infty.$$

Therefore, $f|_{\mathbb{R}^2_+} \in H^{1,1}_{r,0}(\mathbb{R}^2_+)$. It remains to show that $f|_{\mathbb{R}^2_+} \notin H^{1,1}_z(\mathbb{R}^2_+)$.

Assume that $\partial_y\left(f|_{\mathbb{R}^2_+}\right) \in H^1_z(\mathbb{R}^2_+)$ and denote by F its extension by zero outside $\mathbb{R}^2_+$. This means that $F \in H^1(\mathbb{R}^2)$. Using the well-known duality between $H^1(\mathbb{R}^2)$ and $\mathrm{BMO}(\mathbb{R}^2)$ (see [14]), one has, for all function $\phi \in L^\infty(\mathbb{R}^2)$,

$$\left|\int_{\mathbb{R}^2} F(x,y)\phi(x,y)\,dx\,dy\right| \leqslant \|F\|_{H^1(\mathbb{R}^2)}\,\|\phi\|_{\mathrm{BMO}(\mathbb{R}^2)},$$

the integral being absolutely convergent. For all $k \in \mathbb{N}^*$ and all $(x, y) \in \mathbb{R}^2$, define

$$\psi_k(x, y) = \begin{cases} \ln |y| \text{ if } e^{-k} \leqslant |y| \leqslant e^k, \\ 0 \qquad \text{otherwise.} \end{cases}$$

Since $(x, y) \mapsto \ln |y| \in \mathrm{BMO}(\mathbb{R}^2)$, there exists $C > 0$ such that, for all k, $\|\psi_k\|_{\mathrm{BMO}(\mathbb{R}^2)} \leqslant C$ (see [25, Chapter 4, p. 143]). Therefore, we obtain, for all k,

$$\left| \int_{x \in \mathbb{R},\ e^{-k} \leqslant y \leqslant e^k} \phi(x) h(y) \ln\, y \, dx \, dy \right| \leqslant C \|F\|_{H^1(\mathbb{R}^2)},$$

which means that

$$\int_{e^{-k}}^{e^k} h(y) \ln\, y \, dy \leqslant C \|F\|_{H^1(\mathbb{R}^2)}.$$

But this is false, since

$$\int_0^{+\infty} h(y) \ln\, y \, dy = +\infty$$

and we get a contradiction.

2.3. A trace theorem

Another feature which distinguishes $H_r^{1,1}$ spaces from $H_z^{1,1}$ spaces is trace properties. It is well-known that $\mathrm{Tr}(\dot{H}_r^{1,1}(\mathbb{R}^n_+)) = L^1(\partial \mathbb{R}^n_+)$ (see [15, Theorem 11.1]). Therefore, $\mathrm{Tr}(\dot{H}_z^{1,1}(\mathbb{R}^n_+))$ must be a subspace of $L^1(\partial \mathbb{R}^n_+)$, even contained in

$$L^1_0(\mathbb{R}^{n-1}) = \left\{ f \in L^1(\mathbb{R}^{n-1});\ \int_{\mathbb{R}^{n-1}} f(x')\, dx' = 0 \right\}$$

(we identify $\partial \mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$). It turns out that it is even smaller. More precisely, one has:

Proposition 4. $\mathrm{Tr}(\dot{H}_z^{1,1}(\mathbb{R}^n_+)) = \dot{B}_1^{0,1}(\partial \mathbb{R}^n_+)$.

Before proving Proposition 4, let us recall that $\mathrm{Tr}(\dot{W}^{1,p}(\mathbb{R}^n_+)) = \dot{B}_p^{1-1/p,p}(\partial \mathbb{R}^n_+)$ for all $1 < p < +\infty$. Also $\dot{W}^{1,p}(\mathbb{R}^n_+)$ may be regarded as the space of restrictions to $\mathbb{R}^n_+$ of functions in $\dot{W}^{1,p}(\mathbb{R}^n)$. Hence, for $p = 1$, it is not the natural candidate $\dot{H}_r^{1,1}(\mathbb{R}^n_+)$ whose trace space is a Besov space, but the strictly smaller space $\dot{H}_z^{1,1}(\mathbb{R}^n_+)$.

We first establish the continuity of Tr. Let $f \in \dot{H}_z^{1,1}(\mathbb{R}_+^n)$. Since $\nabla f \in H_z^1(\mathbb{R}_+^n)$, its extension by 0, say $\mathbf{F}$, to $\mathbb{R}^n$, belongs to $H^1(\mathbb{R}^n)$. Hence, by the Littlewood–Paley characterization of $H^1(\mathbb{R}^n)$ (see [30]), for any $\mathbf{\Phi} = (\Phi_1, \dots, \Phi_n) \in \mathcal{D}(\mathbb{R}^n, \mathbb{R}^n)$ with

$$0 < \inf_{\xi \neq 0} \int_0^{+\infty} \left|\widehat{\mathbf{\Phi}_i}(t\xi)\right|^2 \frac{dt}{t} \leqslant \sup_{\xi \neq 0} \int_0^{+\infty} \left|\widehat{\mathbf{\Phi}_i}(t\xi)\right|^2 \frac{dt}{t} < +\infty,$$

one has

$$\int_{\mathbb{R}^n} \left(\int_0^{+\infty} |\mathbf{F} *_n \mathbf{\Phi}_t(x)|^2 \frac{dt}{t} \right)^{1/2} dx \sim \|\mathbf{F}\|_{H^1(\mathbb{R}^n)},$$

where $*_k$ denotes the convolution in $\mathbb{R}^k$, $\mathbf{\Phi}_t(x) = \frac{1}{t^n}\mathbf{\Phi}\left(\frac{x}{t}\right)$ and $\mathbf{F} *_n \mathbf{G} = (F_i *_n G_i)_{1 \leqslant i \leqslant n}$ for $\mathbb{C}^n$-valued functions $\mathbf{F}$ and $\mathbf{G}$. If one drops the condition on the infimum for $\mathbf{\Phi}$, only the inequality

$$\int_{\mathbb{R}^n} \left(\int_0^{+\infty} |\mathbf{F} *_n \mathbf{\Phi}_t(x)|^2 \frac{dt}{t} \right)^{1/2} dx \leqslant C \|\mathbf{F}\|_{H^1(\mathbb{R}^n)}$$

holds.

By the Littlewood–Paley characterization of $\dot{B}_1^{0,1}(\mathbb{R}^{n-1})$, we have to show that

$$\int_{\mathbb{R}^{n-1}} \int_0^{+\infty} \left|\mathrm{Tr}\, f *_{n-1} \psi_t(x)\right| \frac{dx\, dt}{t} \leqslant C \|\mathbf{F}\|_{H^1(\mathbb{R}^n)}$$

for some function $\psi \in \mathcal{D}(\mathbb{R}^{n-1})$ with

$$0 < \inf_{\xi \neq 0} \int_0^{+\infty} \left|\widehat{\psi}(t\xi)\right|^2 \frac{dt}{t} \leqslant \sup_{\xi \neq 0} \int_0^{+\infty} \left|\widehat{\psi}(t\xi)\right|^2 \frac{dt}{t} < +\infty$$

and $\psi_t(x) = \frac{1}{t^{n-1}}\psi\left(\frac{x}{t}\right)$. Notice that, since $f \in \dot{H}_z^{1,1}(\mathbb{R}_+^n)$, then $f \in L_c^1(\mathbb{R}_+^n)$, $\nabla f \in L^1(\mathbb{R}_+^n)$ and $\mathrm{Tr}\, f \in L_{\mathrm{loc}}^1(\mathbb{R}^{n-1})$, so that all the computations to follow are justified.

Pick a function $\phi \in \mathcal{D}(\mathbb{R})$ equal to 1 on $[-1,1]$ and supported in $[-2,2]$. For all function $\psi \in \mathcal{D}(\mathbb{R}^{n-1})$, all $t>0$, all $x \in \mathbb{R}^{n-1}$ and all $a \in [-t,t]$,

$$\begin{aligned}
\mathrm{Tr}\, f *_{n-1} \psi_t(x) &= \int_{\mathbb{R}^{n-1}} (\mathrm{Tr}\, f)(y) \frac{1}{t^{n-1}} \psi\left(\frac{x-y}{t}\right) dy \\
&= \int_{\mathbb{R}^{n-1}} (\mathrm{Tr}\, f)(y) \phi\left(\frac{a}{t}\right) \frac{1}{t^{n-1}} \psi\left(\frac{x-y}{t}\right) dy \\
&= -\int_{\mathbb{R}^{n-1}} \left(\int_0^{+\infty} \partial_w \left(f(y,w) \phi\left(\frac{a-w}{t}\right)\right) dw \right) \\
&\quad \times \frac{1}{t^{n-1}} \psi\left(\frac{x-y}{t}\right) dy \\
&= -t \int_{\mathbb{R}^n_+} (\partial_n f)(y,w) \frac{1}{t^n} \phi\left(\frac{a-w}{t}\right) \psi\left(\frac{x-y}{t}\right) dy\, dw \\
&\quad + \int_{\mathbb{R}^n_+} f(y,w) \frac{1}{t^n} \phi'\left(\frac{a-w}{t}\right) \psi\left(\frac{x-y}{t}\right) dy\, dw.
\end{aligned} \tag{1}$$

Fix a nonzero radial function $g \in \mathcal{D}(\mathbb{R}^{n-1})$ and define $\psi = \Delta g$ and, for all $1 \leqslant k \leqslant n-1$, $\psi_k = \partial_k g$, so that $\psi = \sum_{k=1}^{n-1} \partial_k \psi_k$. Observe that

$$\int_0^{+\infty} t^4 \left|\widehat{g}(t\xi)\right|^2 \frac{dt}{t} = c \neq 0$$

does not depend on $\xi \in S^{n-2}$. It follows that ψ satisfies the required conditions. Observe also that, for all $1 \leqslant k \leqslant n-1$, there exists $C_k > 0$ such that, for all $\xi \neq 0$,

$$\int_0^{+\infty} \left|\widehat{\psi_k}(t\xi)\right|^2 \frac{dt}{t} \leqslant C_k.$$

We may integrate by parts in the last integral of (1) with respect to kth variable and obtain, for all $x \in \mathbb{R}^{n-1}$, all $t>0$ and all $a \in [-t,t]$,

$$(\mathrm{Tr}\, f *_{n-1} \psi_t)(x) = -t \left(\mathbf{F}_n *_n \Phi_t^{(n)} \right)(x,a) + \sum_{k=1}^{n-1} t \left(\mathbf{F}_k *_n \Phi_t^{(k)} \right)(x,a),$$

where $\Phi^{(n)} = \psi \otimes \phi$ and $\Phi^{(k)} = \psi_k \otimes \phi'$. As a consequence, if $\mathbf{\Phi} = \left(\Phi^{(1)}, \ldots, \Phi^{(n-1)}, \Phi^{(n)}\right)$, one has, by averaging,

$$\left|\left(\mathrm{Tr}\, f *_{(n-1)} \psi_t\right)(x)\right| \leqslant C \int_{\frac{t}{2} \leqslant |a| \leqslant t} |(\mathbf{F} *_n \mathbf{\Phi}_t)(x, a)| \, da.$$

Integrating over $\mathbb{R}^{n-1} \times]0, +\infty[$ against $\dfrac{dx\,dt}{t}$ yields, by Fubini,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \int_0^{+\infty} \left|\mathrm{Tr}\, f *_{n-1} \psi_t(x)\right| \frac{dx\,dt}{t} \\
&\quad \leqslant C \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{|a|}^{2|a|} |(\mathbf{F} *_n \mathbf{\Phi}_t)(x, a)| \frac{dt}{t} \, dx\, da \\
&\quad \leqslant C (\ln 2)^{1/2} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left(\int_0^{+\infty} |(\mathbf{F} *_n \mathbf{\Phi}_t)(x, a)|^2 \frac{dt}{t} \right)^{1/2} dx\, da,
\end{aligned}$$

which proves the boundedness of Tr.

We now prove that Tr is onto. We use a wavelet basis to compute the norm in $\dot{B}_1^{0,1}(\mathbb{R}^{n-1})$. Let ψ_l, $l \in E = \left\{1, \ldots, 2^{n-1} - 1\right\}$ be a collection of C^1 compactly supported functions such that $\left\{2^{(n-1)j}\psi_l(2^j x - k)\right\}_{j\in\mathbb{Z},\ k\in\mathbb{Z}^{n-1},\ l\in E}$ is an unconditional basis of $\dot{B}_1^{0,1}(\mathbb{R}^{n-1})$ (see [21, Chapter 6]), that is

$$g = \sum_{j,k,l} \lambda_{j,k,l} 2^{(n-1)j} \psi_l(2^j x - k) \in \dot{B}_1^{0,1}(\mathbb{R}^{n-1}) \text{ iff } \sum_{j,k,l} \left|\lambda_{j,k,l}\right| < +\infty$$

and

$$\|g\|_{\dot{B}_1^{0,1}(\mathbb{R}^{n-1})} \sim \sum_{j,k,l} \left|\lambda_{j,k,l}\right|.$$

Note in particular that, for all $l \in E$,

$$\int_{\mathbb{R}^{n-1}} \psi_l(x_1, \ldots, x_{n-1})\, dx_1 \ldots dx_{n-1} = 0.$$

We drop the index l which plays no role.

Let $g = \sum_{j,k} \lambda_{j,k} 2^{(n-1)j} \psi(2^j x - k) \in \dot{B}_1^{0,1}(\mathbb{R}^{n-1})$ with $\sum_{j,k} \left|\lambda_{j,k}\right| \leqslant C \|g\|_{\dot{B}_1^{0,1}(\mathbb{R}^{n-1})}$ and ϕ be the function used in the previous argument. Define, for all $x \in \mathbb{R}^{n-1}$

and all $a > 0$,

$$f(x,a) = \sum_{j,k} \lambda_{j,k} 2^{j(n-1)} \psi(2^j x - k)\phi(2^j a).$$

Clearly, $f \in L^1_c(\mathbb{R}^n_+)$. One has, for all $a > 0$,

$$\nabla_x f(x,a) = \sum_{j,k} \lambda_{j,k} a_{j,k}(x,a),$$

where, for all $(x,a) \in \mathbb{R}^n$,

$$a_{j,k}(x,a) = 2^{nj}(\nabla_x \psi)(2^j x - k)\phi(2^j a)\mathbb{1}_{]0,+\infty[}(a).$$

Then, $a_{j,k}$ is supported in a cube of $\mathbb{R}^n$ with side length $C2^{-j}$ (where $C > 0$ only depends on ψ and ϕ), and satisfies

$$\left\| a_{j,k} \right\|_\infty \leqslant C 2^{nj} \quad \text{and} \quad \int a_{j,k}(x,a)\, dx\, da = 0.$$

Since $a_{j,k}$ is supported in $\overline{\mathbb{R}^n_+}$, we conclude that there exists $C > 0$ such that, for all j, k,

$$\left\| a_{j,k} \right\|_{H^1_z(\mathbb{R}^n_+)} \leqslant C.$$

Therefore, the l^1 condition on the $\lambda_{j,k}$'s yields

$$\| \nabla_x f(x,a) \|_{H^1_z(\mathbb{R}^n_+)} \leqslant C \, \| g \|_{\dot{B}^{0,1}_1(\mathbb{R}^{n-1})}.$$

Next,

$$\partial_a f(x,a) = \sum_{j,k} \lambda_{j,k} b_{j,k}(x,a),$$

where

$$b_{j,k}(x,a) = 2^{nj} \psi(2^j x - k)\phi'(2^j a)\mathbb{1}_{]0,+\infty[}(a).$$

A similar argument shows that

$$\left\| b_{j,k} \right\|_{H^1_z(\mathbb{R}^n_+)} \leqslant C,$$

where $C > 0$ does not depend on j, k (here, the fact that $b_{j,k}$ has zero integral follows from $\int_{\mathbb{R}^{n-1}} \psi = 0$). We conclude that

$$\left\| \partial_a f(x, a) \right\|_{H_z^1(\mathbb{R}_+^n)} \leqslant C \left\| g \right\|_{\dot{B}_1^{0,1}(\mathbb{R}^{n-1})},$$

which ends the proof of Proposition 4.

Remark 5. The extension from $\dot{B}_1^{0,1}(\partial\mathbb{R}_+^n)$ to $\dot{H}_z^{1,1}(\mathbb{R}_+^n)$ is linear, while the one from $L^1(\partial\mathbb{R}_+^n)$ to $\dot{H}_r^{1,1}(\mathbb{R}_+^n)$ is not.

3. Characterization of Hardy–Sobolev spaces through adapted maximal functions

3.1. Global Hardy–Sobolev spaces

Our main task is to represent the norm of ∇f in $H_r^1(\Omega)$ or in $H_z^1(\Omega)$ in terms of appropriate maximal functions. This will be possible only on $H_r^{1,1}(\Omega)$ and $H_{z,0}^{1,1}(\Omega)$.

It is well-known that $H_r^1(\Omega)$ and $H_z^1(\Omega)$ have characterizations through grand maximal functions, which take the following form on strongly Lipschitz domains: if $f \in L_c^1(\Omega)$ and $x \in \Omega$, define

$$M_z f(x) = \sup \left| \int_\Omega f(y)\phi(y)\, dy \right|$$

taken over all C^∞ functions ϕ supported in a cube Q centered in Ω and containing x, and satisfying $\left\|\phi\right\|_\infty + l_Q \left\|\nabla\phi\right\|_\infty \leqslant |Q|^{-1}$; define also

$$M_r f(x) = \sup \left| \int_\Omega f(y)\phi(y)\, dy \right|$$

taken over all C^∞ functions ϕ supported in a cube Q centered in Ω and containing x, satisfying $\left\|\phi\right\|_\infty + l_Q \left\|\nabla\phi\right\|_\infty \leqslant |Q|^{-1}$ and which are zero on $\partial\Omega$. Then, one has (see [9,18])

$$\|f\|_{H_z^1(\Omega)} \sim \|M_z f\|_{L^1(\Omega)} \quad \text{and} \quad \|f\|_{H_r^1(\Omega)} \sim \|M_r f\|_{L^1(\Omega)}.$$

A straightforward consequence of these scalar characterizations is the following vector-valued extension: define $\widetilde{F}_x(\Omega)$ as the space of all functions $\mathbf{\Phi} = (\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n) \in C^\infty(\mathbb{R}^n, \mathbb{C}^n)$ supported in a cube Q of $\mathbb{R}^n$ centered in Ω and containing x, with

$$\|\mathbf{\Phi}\|_\infty + l_Q \|D\mathbf{\Phi}\|_\infty \leqslant \frac{1}{|Q|},$$

where, for all $x \in \mathbb{R}^n$, $D\mathbf{\Phi}(x)$ is the $n \times n$ matrix whose jth column is $\nabla \mathbf{\Phi}_j(x)$. Define also

$$\widetilde{G}_x(\Omega) = \left\{ \mathbf{\Phi} \in \widetilde{F}_x(\Omega);\ \ \mathbf{\Phi} = 0 \text{ on } \partial\Omega \right\}.$$

For all function $\mathbf{F} \in L^1_c(\Omega, \mathbb{C}^n)$, define

$$\widetilde{M}\mathbf{F}(x) = \sup_{\mathbf{\Phi} \in \widetilde{F}_x(\Omega)} |\langle \mathbf{F}, \mathbf{\Phi} \rangle|$$

and

$$\widetilde{N}\mathbf{F}(x) = \sup_{\mathbf{\Phi} \in \widetilde{G}_x(\Omega)} |\langle \mathbf{F}, \mathbf{\Phi} \rangle|,$$

where, for suitable $\mathbb{C}^n$-valued functions $\mathbf{G_1}$ and $\mathbf{G_2}$, we write

$$\langle \mathbf{G_1}, \mathbf{G_2} \rangle = \int_\Omega \mathbf{G_1}(x).\overline{\mathbf{G_2}(x)}\, dx.$$

Then, we have

$$\|\mathbf{F}\|_{H^1_r(\Omega)} \sim \left\| \widetilde{N}\mathbf{F} \right\|_{L^1(\Omega)}$$

and

$$\|\mathbf{F}\|_{H^1_z(\Omega)} \sim \left\| \widetilde{M}\mathbf{F} \right\|_{L^1(\Omega)}.$$

Our main theorem deals with those vector-valued functions $\mathbf{F}$ which are the gradient of some function f. The differential structure of ∇f has consequences on the characterization of such functions via maximal functions. The "formal" integration by parts:

$$\langle \nabla f, \mathbf{\Phi} \rangle = -\int_\Omega f(x) \operatorname{div} \mathbf{\Phi}(x)\, dx$$

suggests that the relevant regularity condition for test functions $\mathbf{\Phi}$, as it will turn out, is to have control over $\|\operatorname{div} \mathbf{\Phi}\|_\infty$ instead of $\|D\mathbf{\Phi}\|_\infty$, and that the second integral should be used to define maximal functions, if one has no a priori knowledge of the differentiability of f.

More precisely, for all $x \in \Omega$, let $F_x(\Omega)$ be the space of all functions $\mathbf{\Phi} = (\mathbf{\Phi}_1, \dots, \mathbf{\Phi}_n) \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$, whose distributional divergence is a bounded function in $\mathbb{R}^n$, supported in a cube Q of $\mathbb{R}^n$ centered in Ω and containing x, with

$$\|\mathbf{\Phi}\|_\infty + l_Q \|\mathrm{div}\, \mathbf{\Phi}\|_\infty \leqslant \frac{1}{|Q|}.$$

For all $x \in \Omega$, define

$$G_x(\Omega) = \left\{ \mathbf{\Phi} \in F_x(\Omega);\ \ \mathbf{\Phi}.\mathbf{v} = 0 \text{ a.e. on } \partial\Omega \right\},$$

where $\mathbf{v}$ denotes the outward unit normal vector. Observe that, when both $\mathbf{\Phi}$ and div $\mathbf{\Phi}$ are bounded, $\mathbf{\Phi}.\mathbf{v}$ is well defined in $L^\infty(\partial\Omega)$.

If $f \in L^1_c(\Omega)$, define, for all $x \in \Omega$,

$$M^{(1)} f(x) = \sup_{\mathbf{\Phi} \in F_x(\Omega)} \left| \int_\Omega f(y)\ \mathrm{div}\, \mathbf{\Phi}(y)\, dy \right|,$$

$$N^{(1)} f(x) = \sup_{\mathbf{\Phi} \in G_x(\Omega)} \left| \int_\Omega f(y)\ \mathrm{div}\, \mathbf{\Phi}(y)\, dy \right|.$$

Then, our main theorem is as follows:

Theorem 6. *Let $f \in L^1_c(\Omega)$. Then,*
(a) *$\nabla f \in H^1_r(\Omega)$ if and only if $N^{(1)} f \in L^1(\Omega)$. Moreover,*

$$\|\nabla f\|_{H^1_r(\Omega)} \sim \left\| N^{(1)} f \right\|_{L^1(\Omega)}.$$

(b) *$\nabla f \in H^1_z(\Omega)$ and f has zero trace on $\partial\Omega$ if and only if $M^{(1)} f \in L^1(\Omega)$. Moreover,*

$$\|\nabla f\|_{H^1_z(\Omega)} \sim \left\| M^{(1)} f \right\|_{L^1(\Omega)}.$$

Proof. We first have to show that, if $N^{(1)} f \in L^1(\Omega)$ (resp. $M^{(1)} f \in L^1(\Omega)$), then $\nabla f \in H^1_r(\Omega)$ (resp. $\nabla f \in H^1_z(\Omega)$ and f has zero trace on $\partial\Omega$).

To that purpose, we need some further notations. For all $x \in \Omega$, say that a complex-valued function $\phi \in \mathcal{D}(\Omega)$ is in $B_x(\Omega)$ if it is supported in a cube $Q \subset \Omega$ containing x and satisfies $\left\|\phi\right\|_\infty \leqslant |Q|^{-1}$ and $l_Q \left\|\nabla\phi\right\|_\infty \leqslant |Q|^{-1}$. Then, the following holds:

Lemma 7. *Let $T \in \mathcal{D}'(\Omega)$. Define*

$$h(x) = \sup_{\phi \in B_x(\Omega)} \left| \langle T, \phi \rangle \right|.$$

Assume that $h \in L^1_{\mathrm{loc}}(\Omega)$. Then T is a function in $L^1_{\mathrm{loc}}(\Omega)$.

This result is classical and we include a quick proof for the convenience of the reader. We first prove that T is a measure. Let ψ be a nonnegative function in $\mathcal{D}(\mathbb{R}^n)$, and for all $\varepsilon > 0$, define

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^n}\psi\left(\frac{x}{\varepsilon}\right).$$

One has $T * \psi_\varepsilon \to T$ in $\mathcal{D}'(\Omega)$. Therefore, if $\phi \in \mathcal{D}(\Omega)$,

$$\langle T, \phi\rangle = \lim_{\varepsilon\to 0}\langle T * \psi_\varepsilon, \phi\rangle.$$

For all $y \in \operatorname{Supp}\phi$ and all sufficiently small ε, $\frac{1}{C}\psi_\varepsilon(y - .) \in B_y(\Omega)$ for some constant $C > 0$ only depending on ψ. It follows that, on $\operatorname{Supp}\phi$,

$$\left|T * \psi_\varepsilon\right| \leqslant Ch.$$

Thus, we obtain that, for all compact set $K \subset \Omega$, there exists $C_K > 0$ such that, for all $\phi \in \mathcal{D}(\Omega)$ supported in K,

$$\left|\langle T, \phi\rangle\right| \leqslant C_K \left\|\phi\right\|_\infty,$$

which means that T is a measure, which will be denoted by μ.

We now check that μ is absolutely continuous with respect to the Lebesgue measure. Consider a compact set $K \subset \Omega$ and a measurable subset $A \subset K$. Using a similar argument, we obtain

$$|\mu(A)| = |\langle \mu, \mathbb{1}_A\rangle| \leqslant C\int_A h(y)\,dy,$$

which ends the proof. $\square$

We derive from Lemma 7 the following result:

Lemma 8. (a) *There exists $C > 0$ such that, for all $f \in L^1_c(\Omega)$ and all $\mathbf{u} \in \mathcal{D}(\Omega, \mathbb{C}^n)$,*

$$\left|\int_\Omega f(x)\operatorname{div}\mathbf{u}(x)\,dx\right| \leqslant C\int_\Omega N^{(1)}f(x)\,|\mathbf{u}(x)|\,dx.$$

(b) *There exists $C > 0$ such that, for all $f \in L^1_c(\Omega)$ and all $\mathbf{u} \in \mathcal{D}(\mathbb{R}^n, \mathbb{C}^n)$,*

$$\left|\int_\Omega f(x)\operatorname{div}\mathbf{u}(x)\,dx\right| \leqslant C\int_\Omega M^{(1)}f(x)\,|\mathbf{u}(x)|\,dx.$$

We begin with the proof of (a). Using that the support of $\mathbf{u}$ is a compact of $\mathbb{R}^n$ included in Ω, one has, for all $1 \leqslant i \leqslant n$ and ψ_ε as in the proof of Lemma 7,

$$\left|\int_\Omega f(x)\partial_i \mathbf{u}_i(x)\,dx\right| = \left|\lim_{\varepsilon\to 0}\int_\Omega f(x)\partial_i(\mathbf{u}_i * \psi_\varepsilon)(x)\,dx\right|$$
$$= \left|\lim_{\varepsilon\to 0}\int_\Omega \left(\int_\Omega f(x)\partial_i\psi_\varepsilon(x-y)\,dx\right)\mathbf{u}_i(y)\,dy\right|$$

We see that the innermost integral in the second line is equal to

$$\int_\Omega f(x)\operatorname{div}\mathbf{\Psi}_\varepsilon(x)\,dx$$

where $\mathbf{\Psi}_\varepsilon(x) = \psi_\varepsilon(x-y)\mathbf{e}_i$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ denotes the canonical basis in $\mathbb{R}^n$. Since the function $\dfrac{1}{C}\,\mathbf{\Psi}_\varepsilon(.-y)$ belongs to $G_y(\Omega)$, where $C > 0$ only depends on ψ, we conclude that

$$\left|\int_\Omega f(x)\partial_i \mathbf{u}_i(x)\,dx\right| \leqslant C\int_\Omega N^{(1)}f(y)\,|\mathbf{u}(y)|\,dy.$$

To prove (b), let $(K_p)_{p\geqslant 0}$ be an exhaustion of Ω by compact subsets of $\mathbb{R}^n$. For each $p \geqslant 0$, let η_p be a C^∞ function supported in K_{p+2}, equal to 1 in K_{p+1} and such that $0 \leqslant \eta_p \leqslant 1$. As before, one has, for all $1 \leqslant i \leqslant n$ and all $p \geqslant 0$,

$$\begin{aligned}
&\left|\int_\Omega f(x)\partial_i(\eta_p \mathbf{u}_i)(x)\,dx\right| \\
&= \left|\lim_{\varepsilon\to 0}\int_\Omega f(x)\partial_i(\eta_p\mathbf{u}_i * \psi_\varepsilon)(x)\,dx\right| \\
&= \left|\lim_{\varepsilon\to 0}\int_\Omega\left(\int_\Omega f(x)\partial_i\psi_\varepsilon(x-y)\,dx\right)(\eta_p\mathbf{u}_i)(y)\,dy\right| \\
&\leqslant C\int_\Omega M^{(1)}f(y)\left|\eta_p(y)\mathbf{u}(y)\right|\,dy \\
&\leqslant C\int_\Omega M^{(1)}f(y)\,|\mathbf{u}(y)|\,dy.
\end{aligned}$$

Letting $p \to +\infty$ yields (b).

We now go back to proving the "if" part in Theorem 6. Assume first that $f \in L^1_c(\Omega)$ and that $N^{(1)}f \in L^1(\Omega)$. Lemma 7, applied to $T = \partial_i f$ shows that $\partial_i f \in L^1_{\mathrm{loc}}(\Omega)$ for all $1 \leqslant i \leqslant n$, since $h \leqslant N^{(1)}f$. Then, assertion (a) in Lemma 8 yields $\nabla f \in L^1(\Omega)$.

Finally, we use the maximal function $\widetilde{N}(\nabla f)$ to show that $\nabla f \in H^1_r(\Omega)$. Indeed, if $\mathbf{\Phi} \in \widetilde{G}_x(\Omega)$, the Green–Riemann formula

$$\int_\Omega f(y)\operatorname{div}\mathbf{\Phi}(y)\,dy = -\langle \nabla f, \mathbf{\Phi}\rangle$$

applies and yields $\widetilde{N}(\nabla f) \leqslant N^{(1)} f$.

Assume now that $f \in L^1_c(\Omega)$ and that $M^{(1)} f \in L^1(\Omega)$. Since $N^{(1)} f \in L^1(\Omega)$, we already know that $\nabla f \in H^1_r(\Omega)$. In particular, $\operatorname{Tr} f$ exists in $L^1(\partial\Omega)$. By the Green–Riemann formula and assertion (b) in Lemma 8, one has, for all $\mathbf{u} \in \mathcal{D}(\mathbb{R}^n, \mathbb{C}^n)$,

$$\left| \int_{\partial\Omega} \operatorname{Tr} f(x)\mathbf{u}(x).\mathbf{v}(x)\,d\sigma(x)\right| \leqslant \int_\Omega h(x)\,|\mathbf{u}(x)|\,dx,$$

where $h = C(M^{(1)} f + |\nabla f|) \in L^1(\Omega)$. Since we can choose $\mathbf{u}$ with $|\operatorname{supp}\mathbf{u}|$ arbitrary small and $\|\mathbf{u}\|_\infty \leqslant 1$, we easily obtain that

$$\int_{\partial\Omega} \operatorname{Tr} f(x)\mathbb{1}_A(x)\,d\sigma(x) = 0,$$

for all measurable sets $A \subset \partial\Omega$, which shows that $\operatorname{Tr}(f) = 0$ on $\partial\Omega$. Hence, for all $\mathbf{\Phi} \in \widetilde{F_x}(\Omega)$,

$$\int_\Omega f(x)\operatorname{div}\mathbf{\Phi}(x)\,dx = -\langle \nabla f, \mathbf{\Phi}\rangle,$$

which yields $\widetilde{M}(\nabla f) \leqslant M^{(1)} f$. Therefore, $\nabla f \in H^1_z(\Omega)$.

We now turn to the converse implications in Theorem 6. The following lemma is the key result to establish them:

Lemma 9. *Let U be a bounded strongly Lipschitz domain of $\mathbb{R}^n$ and $1 < p < +\infty$. Then there exists $C > 0$ only depending on p and Lipschitz constants of U such that, for all $f \in L^p(U)$,*

$$\|f\|_{L^p(U)} \leqslant C\left(\|f\|_{W^{-1,p}(U)} + \|\nabla f\|_{W^{-1,p}(U)}\right).$$

This result is due to Necas when $p = 2$ (see [23, Chapitre 3, Lemme 7.1]), and the proof is completely identical when $1 < p < +\infty$. As a consequence, we derive the following:

Lemma 10. *Let U be a bounded strongly Lipschitz domain of $\mathbb{R}^n$ and $1 < p < +\infty$. If $f \in L^p(U)$ has zero integral, then there exists $\mathbf{F} \in W^{1,p}_0(U; \mathbb{C}^n)$ such that $f = \operatorname{div}\mathbf{F}$,* *and* $\|D\mathbf{F}\|_p \leqslant C\,\|f\|_p$. *In this statement,* $C>0$ *only depends on* p *and the Lipschitz constants of* U.

Define the operator T from $L^{p'}(U)$ to $W^{-1,p'}(U;\mathbb{C}^n)$ by $T(f)=\nabla f$. Lemma 9 and the compactness of the embedding $L^{p'}(U)\hookrightarrow W^{-1,p'}(U;\mathbb{C}^n)$ imply that T is a closed operator (see [28, Proposition 6.7, p. 498]). It follows that the range of T' is equal to $(\ker T)^{\perp}$. But T' is the operator from $W_0^{1,p}(U;\mathbb{C}^n)$ to $L^p(U)$ given by $T'(\mathbf{F})=-\operatorname{div}\mathbf{F}$, and $(\ker T)^{\perp}$ is exactly the space of all functions in $L^p(U)$ which have zero integral.

We now define L^p variants of $\widetilde{M}$ and $\widetilde{N}$. For all $x\in\Omega$ and all $1<p<+\infty$, say that a function $\mathbf{\Phi}\in W^{1,p}(\mathbb{R}^n,\mathbb{C}^n)$ belongs to $\widetilde{F}_{p,x}(\Omega)$ if it is supported in a cube Q centered in Ω and containing x, with

$$\|\mathbf{\Phi}\|_p + l_Q\,\|D\mathbf{\Phi}\|_p \leqslant |Q|^{-1/p'},$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Say that $\mathbf{\Phi}$ belongs to $\widetilde{G}_{p,x}(\Omega)$ if it belongs to $\widetilde{F}_{p,x}(\Omega)$ and has zero trace on $\partial\Omega$.

If $\mathbf{F}\in L^1_c(\Omega,\mathbb{C}^n)$, define, for all $x\in\Omega$,

$$\widetilde{M}_p(\mathbf{F})(x)=\sup_{\mathbf{\Phi}\in\widetilde{F}_{p,x}(\Omega)}|\langle\mathbf{F},\mathbf{\Phi}\rangle|,$$

$$\widetilde{N}_p(\mathbf{F})(x)=\sup_{\mathbf{\Phi}\in\widetilde{G}_{p,x}(\Omega)}|\langle\mathbf{F},\mathbf{\Phi}\rangle|.$$

The last lemma used in the proof of Theorem 6 is the following one:

Lemma 11. *If* $p>2n$, *then there exists* $C>0$ *such that*
(a) $\left\|\widetilde{N}_p(\mathbf{F})\right\|_1 \leqslant C\,\|\mathbf{F}\|_{H^1_r(\Omega)}$ *for all* $\mathbf{F}\in H^1_r(\Omega,\mathbb{C}^n)$,
(b) $\left\|\widetilde{M}_p(\mathbf{F})\right\|_1 \leqslant C\,\|\mathbf{F}\|_{H^1_z(\Omega)}$ *for all* $\mathbf{F}\in H^1_z(\Omega,\mathbb{C}^n)$.

The definitions of $\|\mathbf{F}\|_{H^1_r(\Omega)}$ and $\|\mathbf{F}\|_{H^1_z(\Omega)}$ show that we can argue with scalar-valued functions. At this point, we use the atomic decomposition for Hardy spaces on Ω, which we briefly recall (see [10,9] for details).

A cube Q is said to be an interior cube [with respect to Ω] if $4Q\subset\Omega$ (it is called a type (a) cube in [10]), a boundary cube if $2Q\subset\Omega$ and $4Q\cap\partial\Omega\neq\emptyset$ (it is called a type (b) cube in [10]).

A measurable function a on Ω is called an interior atom (a type (a) atom in [10]) if it is supported in an interior cube Q with

$$\int_Q a(x)\,dx=0 \text{ and } \|a\|_2\leqslant|Q|^{-1/2}.$$

A measurable function a on Ω is called a boundary atom (a type (b) atom in [10]) if it is supported in a boundary cube Q with

$$\|a\|_2 \leqslant |Q|^{-1/2}.$$

Note that a boundary atom is not supposed to have mean value zero.

If a function f defined on Ω belongs to $H^1_r(\Omega)$, then f may be decomposed as

$$f = \sum_{\text{interior}} \lambda_Q a_Q + \sum_{\text{boundary}} \mu_Q b_Q,$$

where the a_Q's are interior atoms, the b_Q's are boundary atoms and $\sum_{\text{interior}} |\lambda_Q| + \sum_{\text{boundary}} |\mu_Q| \leqslant C \|f\|_{H^1_r(\Omega)}$. If Ω^c is unbounded, the converse is also true.

A function f defined on Ω belongs to $H^1_z(\Omega)$ if and only if

$$f = \sum_{\text{interior}} \lambda_Q a_Q,$$

where the a_Q's are interior atoms and $\sum_{\text{interior}} |\lambda_Q| < +\infty$ and $\|f\|_{H^1_z(\Omega)}$ is equivalent to the infimum of $\sum_{\text{interior}} |\lambda_Q|$ over all such decompositions. Actually, in [10], the interior atoms for $H^1_z(\Omega)$ are just supported in Ω, but an examination of the proof shows this more precise statement on a strongly Lipschitz domain (see also [9, Remark 2, p. 1612]).

Observe that, in these decompositions, cubes may be replaced by balls.

For the proof of Lemma 11, we therefore assume that a is a suitable atom and prove the corresponding inequality for a.

Assume first that a is an interior atom. It is supported in a ball $B = B(x_B, r_B) \subset \Omega$ such that $4B \subset \Omega$, with $\|a\|_2 \leqslant |B|^{-1/2}$ and $\int a = 0$. Notice that, for all functions $f \in L^1_c(\Omega)$ and all $x \in \Omega$,

$$\widetilde{M}_p f(x) \leqslant \left(M_{\mathrm{HL}} |f|^{p'} \right)^{1/p'} (x),$$

where M_{HL} denotes the usual Hardy–Littlewood maximal function on $\mathbb{R}^n$ and f is meant to be 0 outside Ω. It follows that

$$\int_{4B} \widetilde{M}_p(a)(x)\, dx \leqslant \left(\int_{\mathbb{R}^n} \left(M_{\mathrm{HL}} |a|^{p'} \right)^{2/p'} (x)\, dx \right)^{1/2} |B|^{1/2}$$

$$\leqslant C \left(\int_{\mathbb{R}^n} |a(x)|^2 \, dx \right)^{1/2} |B|^{1/2}$$

$$\leqslant C.$$

The second line uses the fact that $p' < 2$.

Consider now $x \notin 4B$, and let $\phi \in \widetilde{F}_{p,x}(\Omega)$, scalar-valued, such that $\int_\Omega a(y)\phi(y)\,dy \neq 0$, so that $l_Q \geqslant \delta\,|x - x_B|$. One has

$$\int_\Omega a(y)\phi(y)\,dy = \int_\Omega a(y) \left(\phi(y) - \phi(x_B) \right) \, dx,$$

so that

$$\begin{aligned}\left| \int_\Omega a(y)\phi(y)\,dy \right| &\leqslant \int_B |a(y)| \left(\int_0^1 \left| \nabla\phi(ty + (1-t)x_B) \right| \, dt \right) |y - x_B| \, dy \\ &\leqslant r_B \int_0^1 \int_B \left| \nabla\phi(ty + (1-t)x_B) \right| |a(y)| \, dt\,dy \\ &= r_B \int_B |a(y)|\, F(y)\,dy,\end{aligned}$$

where

$$F(y) = \int_0^1 \left| \nabla\phi(ty + (1-t)x_B) \right| \, dt.$$

For each $0 < t < 1$, one has

$$\int_B \left| \nabla\phi(ty + (1-t)x_B) \right|^2 \, dx = \frac{1}{t^n} \int_{tB} \left| \nabla\phi(z) \right|^2 \, dz,$$

so that

$$\begin{aligned}\int_B F^2(y)\,dy &\leqslant \int_B \int_0^1 \left| \nabla\phi(ty + (1-t)x_B) \right|^2 \, dy\,dt \\ &\leqslant \int_0^1 \left(\int_{tB} \left| \nabla\phi(z) \right|^2 \, dz \right) \frac{dt}{t^n} \\ &\leqslant \int_B \left| \nabla\phi(z) \right|^2 \left(\int_{|z - x_B|/r_B}^1 \frac{dt}{t^n} \right) dz\end{aligned}$$

$$
\begin{aligned}
&= C \int_B \left|\nabla\phi(z)\right|^2 \frac{r_B^{n-1}}{|z-x_B|^{n-1}}\,dz \\
&\leqslant C\left(\int_B \frac{1}{|z-x_B|^{(n-1)q'}}\,dy\right)^{1/q'} \frac{1}{l_Q^{2+2n/p'}} r_B^{n-1} \\
&= C\,\frac{r_B^{n/q'}}{l_Q^{2+2n/p'}},
\end{aligned}
$$

where $\frac{2}{p}+\frac{1}{q'}=1$. The last line holds because $p>2n$. Thus,

$$
\begin{aligned}
\left|\int_\Omega a(y)\phi(y)\,dy\right| &\leqslant r_B\,\|a\|_2\,\|F\|_2 \\
&\leqslant \frac{1}{r_B^n}\left(\frac{r_B}{l_Q}\right)^{1+n/p'} \\
&\leqslant C\,\frac{1}{|B|}\left(\frac{r_B}{|x-x_B|}\right)^{1+n/p'},
\end{aligned}
$$

where we used the fact that $l_Q \geqslant \delta\,|x-x_B|$. Since $1+\frac{n}{p'}>n$, one obtains

$$
\int_{x\notin 4B} \widetilde{M_p}(a)(x)\,dx \leqslant C.
$$

This proves (b).

Assume now that a is supported in a ball B such that $2B\subset\Omega$ but $4B\cap\Omega^c\neq\emptyset$ with $\|a\|_2\leqslant|B|^{-1/2}$. Let $\phi\in\widetilde{G}_{p,x}(\Omega)$. Argue as before with $\int_{4B}\widetilde{N}_p a(x)\,dx$. If $x\notin 4B$ and $\int a(y)\phi(y)\,dy\neq 0$, do analogous computations, replacing x_B by a point $x'_B\in\partial\Omega$ such that $\left|x_B-x'_B\right|\sim r_B$. Therefore, (a) also holds.

Again, it should be noted that, in the definitions of $\widetilde{M}_p$ and $\widetilde{N}_p$, the function $\mathbf{F}$ is an arbitrary vector-valued function (compare with Theorem 6).

We now finish the proof of Theorem 6. Assume that $\nabla f\in H_r^1(\Omega)$, and let $x\in\Omega$ and $\mathbf{\Phi}\in G_x(\Omega)$. By Lemma 10 applied to $\operatorname{div}\mathbf{\Phi}$ on $U=Q\cap\Omega$ with an exponent $p>2n$, there exists $\widetilde{\mathbf{\Phi}}\in W_0^{1,p}(Q\cap\Omega)$ such that $\operatorname{div}\widetilde{\mathbf{\Phi}}=\operatorname{div}\mathbf{\Phi}$ and $\left\|D\widetilde{\mathbf{\Phi}}\right\|_p\leqslant C\,\|\operatorname{div}\mathbf{\Phi}\|_p\leqslant C\,\frac{1}{l_Q}\,|Q|^{-1/p'}$. The Poincaré inequality therefore yields that $\frac{1}{C}\,\widetilde{\mathbf{\Phi}}\in\widetilde{G}_{p,x}(\Omega)$ for a constant $C > 0$ independent from $\mathbf{\Phi}$ and x. Moreover,

$$\int_\Omega f(y) \mathrm{div}\, \mathbf{\Phi}(y)\, dy = \int_\Omega f(y) \mathrm{div}\, \widetilde{\mathbf{\Phi}}(y)\, dy = -\langle \nabla f, \widetilde{\mathbf{\Phi}} \rangle.$$

It follows that $N^{(1)} f \leqslant C \widetilde{N_p}(\nabla f)$, which ends the proof of (a).

The proof of (b) is similar. This time, apply Lemma 10 on $U = Q$ instead of $Q \cap \Omega$. Since $\mathrm{Tr}\, f = 0$ on $\partial\Omega$, one always has $\int_\Omega f\, \mathrm{div}\, \mathbf{\Phi} = \int_\Omega f\, \mathrm{div}\, \widetilde{\mathbf{\Phi}} = -\langle \nabla f, \widetilde{\mathbf{\Phi}} \rangle$. Theorem 6 is proved.

Remark 12. The conclusions of Theorem 6 hold when $\Omega = \mathbb{R}^n$. In this case, $H^{1,1}_{z,0}$ and $H^{1,1}_r$ are identical.

3.2. Local Hardy–Sobolev spaces

The previous characterizations can be adapted to the case of local Hardy–Sobolev spaces. Recall that a locally integrable function f on $\mathbb{R}^n$ belongs to $h^1(\mathbb{R}^n)$ if and only if

$$\left(\sup_{0<t<1} \left| \phi_t * f \right| \right) \in L^1(\mathbb{R}^n),$$

where ϕ is a fixed nonnegative function in $\mathcal{D}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi = 1$ and $\phi_t(x) = \frac{1}{t^n} \phi(x/t)$ (see [16]). Notice that $H^1(\mathbb{R}^n) \subset h^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and that a function $f \in h^1(\mathbb{R}^n)$ does not necessarily have zero integral. Moreover, if $f \in h^1(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$, then $\phi f \in h^1(\mathbb{R}^n)$.

Replacing $H^1(\mathbb{R}^n)$ by $h^1(\mathbb{R}^n)$, one defines $h^1_r(\Omega)$ and $h^1_z(\Omega)$. Mimicking the definitions of "global" Hardy–Sobolev spaces, one defines $h^{1,1}_r(\Omega), h^{1,1}_{r,0}(\Omega), h^{1,1}_z(\Omega)$ and $h^{1,1}_{z,0}(\Omega)$.

The first and last of these "local" Hardy–Sobolev spaces can be characterized in terms of "local" maximal functions. Let $\delta > 0$. For $x \in \Omega$, denote by $F^{\mathrm{loc}}_x(\Omega)$ the class of all functions $\mathbf{\Phi} = (\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_n) \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$, whose distributional divergence is a bounded function in $\mathbb{R}^n$, supported in a cube Q of $\mathbb{R}^n$ centered in Ω, containing x and satisfying $l_Q \leqslant \delta$, with

$$\|\mathbf{\Phi}\|_\infty + l_Q \|\mathrm{div}\, \mathbf{\Phi}\|_\infty \leqslant \frac{1}{|Q|}.$$

For all $x \in \Omega$, define

$$G^{\mathrm{loc}}_x(\Omega) = \left\{ \mathbf{\Phi} \in F^{\mathrm{loc}}_x(\Omega);\ \ \mathbf{\Phi}.\mathbf{\nu} = 0 \text{ a.e. on } \partial\Omega \right\},$$

For $f \in L^1_c(\Omega)$ and $x \in \Omega$, define

$$M^{(1)}_{\text{loc}} f(x) = \sup_{\mathbf{\Phi} \in F^{\text{loc}}_x} \left| \int_\Omega f(y) \operatorname{div} \mathbf{\Phi}(y)\, dy \right|$$

and

$$N^{(1)}_{\text{loc}} f(x) = \sup_{\mathbf{\Phi} \in G^{\text{loc}}_x} \left| \int_\Omega f(y) \operatorname{div} \mathbf{\Phi}(y)\, dy \right|.$$

Then, the local version of Theorem 6 is as follows:

Theorem 13. *Let $f \in L^1_c(\Omega)$. Then,*

(a) $\nabla f \in h^1_r(\Omega)$ *if and only if* $N^{(1)}_{\text{loc}} f \in L^1(\Omega)$. *Moreover,*

$$\|\nabla f\|_{h^1_r(\Omega)} \sim \left\| N^{(1)}_{\text{loc}} f \right\|_{L^1(\Omega)}.$$

(b) $\nabla f \in h^1_z(\Omega)$ *and f has zero trace on $\partial\Omega$ if and only if* $M^{(1)}_{\text{loc}} f \in L^1(\Omega)$. *Moreover,*

$$\|\nabla f\|_{h^1_z(\Omega)} \sim \left\| M^{(1)}_{\text{loc}} f \right\|_{L^1(\Omega)}.$$

The proof is entirely similar to the global case. One just has to use the atomic decomposition for local Hardy spaces (see [10,9]). It is easy to see that different choices for δ in the definition of local test functions lead to the same local Hardy–Sobolev spaces, as long as the class $F^{\text{loc}}_x(\Omega)$ is not empty.

We conclude this section by giving the link between local and global Hardy–Sobolev spaces:

Lemma 14. *There exists $C > 0$ such that, for all $f \in L^1(\Omega)$ with $\nabla f \in h^1(\Omega)$,*

$$\|\nabla f\|_{H^1_r(\Omega)} \leqslant C \left(\|f\|_1 + \|\nabla f\|_{h^1_r(\Omega)} \right)$$
$$\left(resp.\ \|\nabla f\|_{H^1_z(\Omega)} \leqslant C \left(\|f\|_1 + \|\nabla f\|_{h^1_z(\Omega)} \right) \right).$$

Fix $\delta = 1$. Let $x \in \Omega$ and $\mathbf{\Phi}$ be a function supported in a cube Q containing x, satisfying

$$\|\mathbf{\Phi}\|_\infty + l_Q \|\operatorname{div} \mathbf{\Phi}\|_\infty \leqslant |Q|^{-1}.$$

Assume that $l_Q > 1$. One has

$$\left|\int_\Omega f(y)\operatorname{div}\mathbf{\Phi}(y)\,dy\right| \leqslant C\int \frac{|f(y)|}{(1+|x-y|)^{n+1}}\,dy$$

$$= h(x).$$

Thus,

$$M^{(1)}f \leqslant C\left(M^{(1)}_{\text{loc}}f + h\right),$$

which shows that

$$\left\|M^{(1)}f\right\|_1 \leqslant C\left(\left\|M^{(1)}_{\text{loc}}f\right\|_1 + \|h\|_1\right) = C'\left(\left\|M^{(1)}_{\text{loc}}f\right\|_1 + \|f\|_1\right).$$

The corresponding inequality for functionals $N^{(1)}$ and $N^{(1)}_{\text{loc}}$ is similar. Lemma 14 is therefore proved.

Remark 15. If Ω is bounded then $H^1_r(\Omega) = h^1_r(\Omega)$ and $h^1_z(\Omega) = H^1_z(\Omega) \oplus \mathbb{C}$ (see [3]). Hence, one has for $f \in L^1_c(\Omega)$, then $\|\nabla f\|_{H^1_*(\Omega)} \sim \|\nabla f\|_{h^1_*(\Omega)}$ where $* = r$ or z.

4. Improved regularity properties

We apply our previous results to obtain div-curl results. The following result is the classical div-curl lemma on $\mathbb{R}^n$ (see [11]):

Proposition 16. *Let* $1 < q, r < +\infty$ *with* $\frac{1}{q} + \frac{1}{r} = 1$. *Let* $f \in \mathcal{D}'(\mathbb{R}^n)$ *such that* $\nabla f \in L^q(\mathbb{R}^n)$ *and* $\mathbf{e} \in L^r(\mathbb{R}^n)$ *such that* $\operatorname{div}\mathbf{e} = 0$ *on* $\mathbb{R}^n$. *Then* $\mathbf{e}.\nabla f \in H^1(\mathbb{R}^n)$.

This result may be extended to domains (see [17]):

Proposition 17. *Let* $1 < q, r < +\infty$ *with* $\frac{1}{q} + \frac{1}{r} = 1$. *Let* $f \in \mathcal{D}'(\Omega)$ *such that* $\nabla f \in L^q(\Omega)$ *and* $\mathbf{e} \in L^r(\Omega)$ *such that* $\operatorname{div}\mathbf{e} = 0$ *in* Ω *and* $\mathbf{e}.\mathbf{v} = 0$ *on* $\partial\Omega$. *Then* $\mathbf{e}.\nabla f \in H^1_z(\Omega)$, *and*

$$\|\mathbf{e}.\nabla f\|_{H^1_z(\Omega)} \leqslant C\,\|\mathbf{e}\|_{L^r(\Omega)}\,\|\nabla f\|_{L^q(\Omega)}.$$

We present a proof of Proposition 17 using maximal functions, adapted from that of [11] in $\mathbb{R}^n$. Let $x \in \Omega$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ supported in a cube Q centered in Ω and containing x, with

$$\|\phi\|_\infty + l_Q \|\nabla\phi\|_\infty \leqslant |Q|^{-1}.$$

Then

$$\int_\Omega \mathbf{e}.(y)\nabla f(y)\phi(y)\,dy = -\int_\Omega (f-\lambda)(y)\,\mathrm{div}\,(\phi\mathbf{e})(y)\,dy$$

$$= -\int_\Omega (f-\lambda)(y)\mathbf{e}(y).\nabla\phi(y)\,dy,$$

where we used the fact that $\mathbf{e}.\mathbf{v}=0$ on $\partial\Omega$ and $\lambda\in\mathbb{C}$ is a constant. Next, choose $\beta>0$ with $\frac{1}{q}-\frac{1}{n}<\frac{1}{\beta}<\frac{1}{q}$ and α such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Using the support and the size conditions on ϕ, we have

$$\begin{aligned}
&\left|\int_\Omega \nabla f(y).\mathbf{e}(y)\phi(y)\,dy\right| \\
&\quad\leqslant \left(\int_{Q\cap\Omega}|\mathbf{e}(y)|^\alpha\,dy\right)^{1/\alpha}\left(\int_{Q\cap\Omega}\left|f(y)-\lambda\right|^\beta\,dy\right)^{1/\beta}\frac{1}{|Q|\,l(Q)} \\
&\quad\leqslant \left(\int_{Q\cap\Omega}|\mathbf{e}(y)|^\alpha\,dy\right)^{1/\alpha}\left(\int_{Q\cap\Omega}|\nabla f(y)|^\gamma\,dy\right)^{1/\gamma}\frac{1}{|Q|\,l(Q)}
\end{aligned}$$

using the Poincaré inequality on $Q\cap\Omega$ with $\frac{1}{\gamma}=\frac{1}{\beta}+\frac{1}{n}>\frac{1}{q}$ with appropriate choice of λ. It follows that

$$\left|\int_\Omega \mathbf{e}.(y)\nabla f(y)\phi(y)\,dy\right| \leqslant C\left(M^\Omega_{\mathrm{HL}}\,|\mathbf{e}|^\alpha\right)^{1/\alpha}(x)\left(M^\Omega_{\mathrm{HL}}\,|\nabla f|^\gamma\right)^{1/\gamma}(x)$$

$$\times\frac{|Q\cap\Omega|^{\frac{1}{\alpha}+\frac{1}{\gamma}}}{|Q|\,l(Q)},$$

with

$$M^\Omega_{\mathrm{HL}} f(x) = \sup\left(\frac{1}{|Q\cap\Omega|}\int_{Q\cap\Omega}|f(y)|\,dy\right)$$

where the supremum is taken over all cubes Q centered in Ω, containing x and such that $l(Q)\leqslant 2\,\mathrm{diam}\,(\Omega)$. Therefore,

$$M_z(\mathbf{e}.\nabla f)(x)\leqslant C\left(M^\Omega_{\mathrm{HL}}\,|\mathbf{e}|^\alpha\right)^{1/\alpha}(x)\left(M^\Omega_{\mathrm{HL}}\,|\nabla f|^\gamma\right)^{1/\gamma}(x)$$

for all $x \in \Omega$. Since $\alpha < r$, $\gamma < q$ and Ω is of homogeneous type, one has

$$\|M_z(\mathbf{e}.\nabla f)\|_1 \leqslant C' \|\mathbf{e}\|_r \|\nabla f\|_q .$$

Proposition 17 follows.

Remark 18. Observe that this argument breaks down if $q = 1$, and in fact, the result is false.

We now give the correct $q = 1$ endpoint version of the div-curl lemma on domains, which is new, even on $\mathbb{R}^n$:

Theorem 19. *Let Ω be a strongly Lipschitz domain and $\mathbf{e} \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ be a vector field with divergence zero, such that $\mathbf{e}.\boldsymbol{\nu} = 0$ on $\partial\Omega$. If $f \in H_r^{1,1}(\Omega)$, then $\mathbf{e}.\nabla f \in H_z^1(\Omega)$ and*

$$\|\mathbf{e}.\nabla f\|_{H_z^1(\Omega)} \leqslant C \|\mathbf{e}\|_\infty \|\nabla f\|_{H_r^1(\Omega)} .$$

Let ϕ as in the proof of Proposition 17. One just has to observe that, for some numerical constant $C > 0$, $\dfrac{1}{C\|\mathbf{e}\|_\infty}\phi\mathbf{e} \in G_x(\Omega)$. It follows by integration by parts that, for all $x \in \Omega$,

$$M_z(\mathbf{e}.\nabla f)(x) \leqslant C \|\mathbf{e}\|_\infty N^{(1)}(f)(x)$$

and Theorem 19 is proved.

If we drop the assumption $\mathbf{e}.\boldsymbol{\nu} = 0$ on $\partial\Omega$, we obtain another version of the div-curl lemma:

Theorem 20. *If $\nabla f \in H_r^1(\Omega)$ and $\mathbf{e} \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ is a vector field with divergence zero in $\mathbb{R}^n$, one has $\mathbf{e}.\nabla f \in H_r^1(\Omega)$ and*

$$\|\mathbf{e}.\nabla f\|_{H_r^1(\Omega)} \leqslant C \|\mathbf{e}\|_\infty \|\nabla f\|_{H_r^1(\Omega)} .$$

If we reinforce the assumption on *f*, we obtain a third version:

Theorem 21. *If $f \in H_{z,0}^{1,1}(\Omega)$ and $\mathbf{e} \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ is a vector field with divergence zero in $\mathbb{R}^n$, one has $\mathbf{e}.\nabla f \in H_z^1(\Omega)$ and*

$$\|\mathbf{e}.\nabla f\|_{H_z^1(\Omega)} \leqslant C \|\mathbf{e}\|_\infty \|\nabla f\|_{H_z^1(\Omega)} .$$

For the proof, take ϕ as before. Since f has zero trace on $\partial\Omega$, one has

$$\int_\Omega \mathbf{e}.(x)\nabla f(y)\phi(y)\,dy = -\int_\Omega f(y)\,\mathrm{div}\,(\phi\mathbf{e})(y)\,dy.$$

Since $\frac{1}{C\|\mathbf{e}\|_\infty}\phi\mathbf{e} \in F_x(\Omega)$ for some $C > 0$, one obtains

$$M_z(\mathbf{e}.\nabla f)(x) \leqslant C\,\|\mathbf{e}\|_\infty\, M^{(1)}(f)(x),$$

and Theorem 21 follows.

Remark 22. As we will see in Theorem 25 below,

$$H^{1,1}_{z,0}(\Omega) = \left\{ f \in L^1(\mathbb{R}^n);\ \ \nabla f \in H^1(\mathbb{R}^n),\ \ \mathrm{supp}\, f \subset \overline{\Omega} \right\}.$$

Thus, Theorem 21 is a consequence of Theorem 19 in the case when $\Omega = \mathbb{R}^n$.

These results have applications to the improvement of regularity for tangential derivatives:

Corollary 23. *Let* $f \in \mathcal{D}'(\mathbb{R}^n_+)$.
(a) *If* $\nabla f \in H^1_r(\mathbb{R}^n_+)$, *then, for all* $1 \leqslant i \leqslant n-1$, $\partial_i f \in H^1_z(\mathbb{R}^n_+)$.
(b) $\|\nabla f\|_{H^1_z(\mathbb{R}^n_+)} \sim \sum_{i=1}^{n-1} \left\|\partial_i f\right\|_{H^1_r(\mathbb{R}^n_+)} + \left\|\partial_n f\right\|_{H^1_z(\mathbb{R}^n_+)}$.
(c) $\|\nabla f\|_{H^1_r(\mathbb{R}^n_+)} \sim \sum_{i=1}^{n-1} \left\|\partial_i f\right\|_{H^1_z(\mathbb{R}^n_+)} + \left\|\partial_n f\right\|_{H^1_r(\mathbb{R}^n_+)}$.

5. Properties of Hardy–Sobolev spaces

5.1. Change of variables

As an application of Theorem 6, we establish that bilipschitz changes of variable operate on Hardy–Sobolev spaces.

Theorem 24. *Let Ω and Ω' be strongly Lipschitz domains and $h : \mathbb{R}^n \to \mathbb{R}^n$ a bilipschitz function such that $h(\Omega') = \Omega$. Let $f \in L^1_c(\Omega)$. One has $\nabla f \in H^1_r(\Omega)$ (resp. $\nabla f \in H^1_z(\Omega)$ and* $\mathrm{Tr}\, f = 0$) *if, and only if, $\nabla(f \circ h) \in H^1_r(\Omega')$ (resp. $\nabla(f\circ h) \in H^1_z(\Omega')$ and* $\mathrm{Tr}(f \circ h) = 0$ on $\partial\Omega'$). *Moreover, one has*

$$\|\nabla f\|_{H^1_r(\Omega)} \sim \|\nabla(f\circ h)\|_{H^1_r(\Omega)}\,,$$
$$(resp.\, \|\nabla f\|_{H^1_z(\Omega)} \sim \|\nabla(f\circ h)\|_{H^1_z(\Omega')})$$

where H^1 stands for H^1_z or H^1_r in both sides.

Before giving the proof, we notice that, if $f \in H^1_z(\Omega)$, $f \circ h$ does not belong to $H^1_z(\Omega')$ in general, since it does not necessarily have zero integral.

Assume that $\nabla(f \circ h) \in H^1_r(\Omega')$ and let $x \in \Omega$. For all $\mathbf{\Phi} \in G_x(\Omega)$, one has

$$\begin{aligned}\int_\Omega f(y) \operatorname{div} \Phi(y)\, dy &= -\int_\Omega \nabla f(y).\mathbf{\Phi}(y)\, dy \\ &= -\int_{\Omega'} \nabla f(h(y)).\mathbf{\Phi}(h(y))\, |\det J_h(y)|\ dy \\ &= \int_{\Omega'} J_h^{-1}(y) \nabla(f \circ h)(y).\mathbf{\Phi}(h(y))\, |\det J_h(y)|\ dy \\ &= \int_{\Omega'} \nabla(f \circ h)(y).(J_h^t)^{-1}(y) \mathbf{\Phi}(h(y))\, |\det J_h(y)|\ dy.\end{aligned}$$

We claim that, if $\mathbf{\Phi}'(z) = \left(J_h^t\right)^{-1}(z)\mathbf{\Phi}(h(z))\, |\det J_h(z)|$, then $\dfrac{1}{C}\mathbf{\Phi}' \in G_y(\Omega')$ for some $C > 0$ only depending on h, if $h(y) = x$. Indeed, $\mathbf{\Phi}'$ is supported in a cube Q' with side length comparable to l_Q (such that $h(Q') \supset Q$). Its L^∞ norm is controlled by l_Q^{-n}. Moreover, for all function $f \in \mathcal{D}(\Omega)$, one has

$$\begin{aligned}\int_\Omega \nabla f(z) \mathbf{\Phi}(z)\, dz &= -\int_\Omega f(z) \operatorname{div} \mathbf{\Phi}(z)\, dz \\ &= -\int_{\Omega'} f(h(z)) (\operatorname{div} \mathbf{\Phi})(h(z))\, |\det J_h(z)|\ dz\end{aligned} \tag{2}$$

and, by the above computation,

$$\int_\Omega \nabla f(z) \mathbf{\Phi}(z)\, dz = -\int_{\Omega'} f(h(z)) \operatorname{div} \mathbf{\Phi}'(z)\, dz. \tag{3}$$

Comparing (2) and (3), one deduces

$$\operatorname{div} \mathbf{\Phi}'(z) = (\operatorname{div} \mathbf{\Phi})\,(h(z))\, |\det J_h(z)|\,.$$

Thus, $\mathbf{\Phi}' \in F_y(\Omega')$. Finally, the Green–Riemann formula yields that $\mathbf{\Phi}'.\mathbf{v}' = 0$ on $\partial\Omega'$. Thus, $\mathbf{\Phi}' \in G_y(\Omega')$, which ends the proof of Theorem 24 for H^1_r. The proof is identical for H^1_z. One just has to observe that $h(\partial\Omega') = \partial\Omega$.

5.2. Extension and restriction theorems

Recall that the homogeneous Triebel–Lizorkin space $\dot{F}_1^{1,2}(\mathbb{R}^n)$ is defined by

$$\dot{F}_1^{1,2}(\mathbb{R}^n) = (-\Delta)^{-1/2}(H^1(\mathbb{R}^n)).$$

In [15,29], this space is defined modulo polynomials. However, it is possible to "realize" this space as a space of distributions, which are functions in $L^{\frac{n}{n-1}}$ by Sobolev embeddings, and this is the definition we choose (see [31,32]). With this definition, $\mathcal{D}(\mathbb{R}^n)$ is dense in $\dot{F}_1^{1,2}(\mathbb{R}^n)$.

If Ω is a strongly Lipschitz domain of $\mathbb{R}^n$, define

$$\dot{F}_{1,r}^{1,2}(\Omega) = \left\{ f \in L^1_c(\Omega);\ \exists F \in \dot{F}_1^{1,2}(\mathbb{R}^n),\quad F|_\Omega = f \right\}$$

equipped with the norm

$$\| f\|_{\dot{F}_{1,r}^{1,2}(\Omega)} = \inf \| F\|_{\dot{F}_1^{1,2}(\mathbb{R}^n)}$$

taken over all functions $F \in \dot{F}_1^{1,2}(\mathbb{R}^n)$ which coincide with f on Ω.

Define

$$\dot{F}_{1,z}^{1,2}(\Omega) = \left\{ f \in \dot{F}_1^{1,2}(\mathbb{R}^n),\ \operatorname{Supp} f \subset \overline{\Omega} \right\},$$

equipped with the norm of $\dot{F}_1^{1,2}(\mathbb{R}^n)$. Then, the extension theorem for Hardy–Sobolev spaces is as follows:

Theorem 25. *Let Ω be a strongly Lipschitz domain.*
(a) $L^1(\Omega) \cap \dot{F}_{1,z}^{1,2}(\Omega) = H_{z,0}^{1,1}(\Omega)$.
(b) $L^1(\Omega) \cap \dot{F}_{1,r}^{1,2}(\Omega) = H_r^{1,1}(\Omega)$.
In the case when Ω is special Lipschitz, as the proof will show, the corresponding equalities for homogeneous spaces hold (*i.e. without the L^1 norms*).

Before we give the proof, we recall that functions in $\dot{F}_1^{1,2}(\mathbb{R}^n)$ have a trace in $L^1(\partial\Omega)$ (see [15, Theorem 11.1]).

We first prove (a). It is plain to see that if $f \in L^1(\Omega)\cap\dot{F}_{1,z}^{1,2}(\Omega)$, then, by the $H^1(\mathbb{R}^n)$-boundedness of the Riesz transforms, $\nabla f \in H^1(\mathbb{R}^n)$, is supported in $\overline{\Omega}$ and f has zero trace on $\partial\Omega$ because $f = 0$ in $\overline{\Omega}^c$. Conversely, let $f \in H_{z,0}^{1,1}(\Omega)$ and F be the extension of f by 0 outside Ω. It is easily checked, using $\operatorname{Tr} f = 0$, that the extension by 0 of ∇f is equal to ∇F, which proves that $\nabla F \in H^1(\mathbb{R}^n)$, hence $(-\Delta)^{1/2} F \in H^1(\mathbb{R}^n)$.

We now turn to the proof of (b). Assume first that $f \in L^1(\Omega)\cap\dot{F}_{1,r}^{1,2}(\Omega)$. There exists $F \in \dot{F}_1^{1,2}(\mathbb{R}^n)$ which coincides with f on Ω. The function ∇F belongs to $H^1(\mathbb{R}^n)$, and its restriction to Ω coincides with ∇f. Therefore, $\nabla f \in H_r^1(\Omega)$, which shows that $f \in H_r^{1,1}(\Omega)$.

Conversely, let $f \in H_r^{1,1}(\Omega)$. We want to extend f to a function $F \in \dot{F}_1^{1,2}(\mathbb{R}^n)$. We consider successively three cases.

The case when $\Omega = \mathbb{R}^n_+$: Denote by F the even extension of f (see Section 2). According to Proposition 2, $\nabla F \in H^1(\mathbb{R}^n)$ and $F \in \dot{F}_1^{1,2}(\mathbb{R}^n)$.

The case when Ω is special Lipschitz: This means that there exists a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega = \left\{(x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n;\ \ x_n > \phi(x_1, \ldots, x_{n-1})\right\}.$$

Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$h(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1}, x_n + \phi(x_1, \ldots, x_{n-1})).$$

One checks at once that h is a bilipschitz map from $\mathbb{R}^n$ onto $\mathbb{R}^n$, and that $h(\mathbb{R}^n_+) = \Omega$. According to Theorem 24, $f \circ h \in H^{1,1}_r(\mathbb{R}^n_+)$. Therefore, there exists $F \in \dot{F}^{1,2}_1(\mathbb{R}^n)$ such that F coincides with $f \circ h$ on $\mathbb{R}^n_+$. If $G = F \circ h^{-1}$, G coincides with f on Ω, and Theorem 24 shows again that $\nabla G \in H^1(\mathbb{R}^n)$.

The case when Ω is strongly Lipschitz: There exist an integer s, a number $d > 0$ and, for $0 \leqslant k \leqslant s$, $C^\infty(\mathbb{R}^n)$ real-valued functions χ_k and η_k such that $\nabla\chi_k$ and $\nabla\eta_k$ have compact supports and open sets O_k, P_k and Ω_k satisfying (see [24]):

(a) $\sum_{0 \leqslant k \leqslant s} \chi_k(x) = 1$ for x in a neighborhood of Ω,
(b) $\Omega_0 = \mathbb{R}^n$, $\operatorname{Supp}\chi_0 \subset O_0 \subset \overline{O_0} \subset P_0 \subset \overline{P_0} \subset \Omega$,
(c) For $k \geqslant 1$, Ω_k is the image of a special Lipschitz domain under an orthogonal transformation in $\mathbb{R}^n$,
(d) For $k \geqslant 1$, O_k and P_k are open neighborhoods of $\operatorname{Supp}\chi_k$ in $\mathbb{R}^n$ such that $\overline{O_k} \subset P_k$, $P_k \cap \Omega = P_k \cap \Omega_k$ and $P_k \cap \partial\Omega = P_k \cap \partial\Omega_k$.
(e) For $k \geqslant 0$, $\operatorname{Supp}\eta_k \subset P_k$, $\eta_k = 1$ on a neighborhood of $\overline{O_k}$, $\eta_k \geqslant 0$ and $\left\|\eta_k\right\|_\infty = 1$,
(f) For $k \geqslant 0$, $d(O_k, P_k^c) \geqslant d$ and $d(\operatorname{Supp}\chi_k, O_k^c) \geqslant d$.

For each k, let $f_k(x) = f(x)\chi_k(x)$ for all $x \in \Omega$. Define the function $\widetilde{f_k}$ on Ω_k by

$$\widetilde{f_k}(x) = \begin{cases} f_k(x) & \text{if } x \in \Omega \cap \Omega_k, \\ 0 & \text{if } x \in \Omega_k \backslash \Omega. \end{cases}$$

We claim that $\nabla\widetilde{f_k} \in H^1_r(\Omega_k)$ and that

$$\left\|\nabla\widetilde{f_k}\right\|_{H^1_r(\Omega_k)} \leqslant C\left(\|\nabla f\|_{H^1_r(\Omega)} + \|f\|_{L^1(\Omega)}\right).$$

In view of Lemma 14, it is enough to show the corresponding estimate for $\left\|\nabla\widetilde{f_k}\right\|_{h^1_r(\Omega_k)}$.

The proof relies on two lemmata. The first one is a consequence of the atomic decomposition for $h^1(\mathbb{R}^n)$ (see [16]):

Lemma 26. *Let $1 < p \leqslant +\infty$. There exists $C > 0$ such that, for all cube $Q \subset \mathbb{R}^n$ and all function $g \in L^p(\mathbb{R}^n)$ supported in Q, $g \in h^1(\mathbb{R}^n)$ and*

$$\|g\|_{h^1(\mathbb{R}^n)} \leqslant C\left(|Q|^{1/p'} + 1\right)\|g\|_p,$$

with $1/p + 1/p' = 1$.

From this, we derive the following result:

Lemma 27. *Let U be a strongly Lipschitz domain of $\mathbb{R}^n$ (or $U = \mathbb{R}^n$), $f \in L^1(U)$ such that $\nabla f \in h^1_r(U)$ and $\chi \in C^\infty(\mathbb{R}^n)$ such that $\nabla\chi$ has compact support. Then $\nabla(f\chi) \in h^1_r(U)$ and*

$$\|\nabla(f\chi)\|_{h^1_r(U)} \leqslant C\left(\|\nabla f\|_{h^1_r(U)} + \|f\|_{L^1(U)}\right).$$

Proof. Write $\nabla f = \chi\nabla f + f\nabla\chi$. Since $\nabla f \in h^1_r(U)$, $\chi\nabla f \in h^1_r(U)$. Moreover, by the Sobolev embeddings, $f \in L^{\frac{n}{n-1}}(U)$ and, since $\nabla\chi$ has compact support,

$$\|f\nabla\chi\|_{L^{\frac{n}{n-1}}(U)} \leqslant C\left(\|\nabla f\|_{h^1_r(U)} + \|f\|_{L^1(U)}\right).$$

But Lemma 26 shows that $f\nabla\chi \in h^1_z(U)$, which ends the proof of Lemma 27. □

Lemma 27 yields

$$\|\nabla f_k\|_{h^1_r(\Omega)} \leqslant C\left(\|\nabla f\|_{H^1_r(\Omega)} + \|f\|_{L^1(\Omega)}\right).$$

By a further restriction, we have that ∇f_k restricted to $\Omega\cap\Omega_k$ belongs to $h^1_r(\Omega\cap\Omega_k)$.

Observe now that

$$\nabla\widetilde{f}_k = \begin{cases} \nabla f_k & \text{on } \Omega\cap\Omega_k, \\ 0 & \text{on } \Omega_k\backslash\Omega. \end{cases}$$

The atomic decomposition for h^1_r spaces (see [10]) yields that for each i, $\partial_i f_k$ has an atomic decomposition on $\Omega\cap\Omega_k$, where the atoms are supported in cubes included in $\Omega\cap\Omega_k$, hence in Ω_k. This atomic decomposition of $\partial_i f_k$ on $\Omega\cap\Omega_k$ is an atomic decomposition of $\partial_i\widetilde{f}_k$ on Ω_k. This shows that $\nabla\widetilde{f}_k \in h^1_r(\Omega_k)$.

In view of the special Lipschitz case, there exists a function $F_k \in L^1(\mathbb{R}^n)\cap\dot{F}^{1,2}_1(\mathbb{R}^n)$ which coincides with $\widetilde{f}_k$ on Ω_k. Define $\widetilde{F}_k = \eta_k F_k$ and

$$\widetilde{F} = \sum_{k=0}^{s} \eta_k F_k.$$

We claim that $\widetilde{F}$ coincides with f on Ω, $\widetilde{F} \in L^1(\mathbb{R}^n)$ and $\nabla\widetilde{F} \in H^1(\mathbb{R}^n)$. First, for each $x \in \Omega$, $k \in \{0,\dots,s\}$, since $\operatorname{Supp}\eta_k \cap \Omega = \operatorname{Supp}\eta_k \cap \Omega_k \cap \Omega$, $\eta_k(x)F_k(x) = \eta_k(x)\widetilde{f}_k(x) = \eta_k(x)\chi_k(x)f(x) = \chi_k(x)f(x)$. Hence $\widetilde{F} = f$ on Ω. Next, it is clear that $\widetilde{F} \in L^1(\mathbb{R}^n)$. Also, by Lemma 27, $\nabla(\eta_k F_k) \in h^1(\mathbb{R}^n)$. Finally, $\widetilde{F} \in L^1(\mathbb{R}^n)$ and $\nabla\widetilde{F} \in h^1(\mathbb{R}^n)$ imply $\nabla\widetilde{F} \in H^1(\mathbb{R}^n)$ by Lemma 14. Theorem 25 is therefore proved.

Remark 28. A consequence of Theorem 25 is that, if $f \in L^1(\Omega)$ and F denotes the extension of f by 0 outside Ω, then $F \in H^{1,1}(\mathbb{R}^n)$ if and only if $f \in H^{1,1}_{z,0}(\Omega)$. Also, there exists $f \in H^{1,1}_{r,0}(\Omega)$ such that its extension by 0 outside Ω does not belong to $H^{1,1}(\mathbb{R}^n)$ (see Proposition 3).

Remark 29. It is known that $\mathbb{1}_\Omega$ is not a pointwise multiplier on $\dot{F}^{1,2}_1(\mathbb{R}^n)$ (see [15, Corollary 13.6], see also [27]), and our result shows that

$$\left\{f \in \dot{F}^{1,2}_1(\mathbb{R}^n);\ \ f\mathbb{1}_\Omega \in \dot{F}^{1,2}_1(\mathbb{R}^n)\right\} = \dot{F}^{1,2}_{1,z}(\Omega) = \dot{H}^{1,1}_{z,0}(\Omega),$$

which gives, in particular, an alternate proof of this fact.

5.3. Dense classes

Theorem 30. (a) *The space of restrictions to Ω of functions in $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^{1,1}_r(\Omega)$.*
(b) *$\mathcal{D}(\Omega)$ is dense in $H^{1,1}_{z,0}(\Omega)$,*
(c) *$\mathcal{D}(\Omega)$ is dense in $H^{1,1}_{r,0}(\Omega)$.*

Proof of (a). Thanks to Theorem 25, (b), it is enough to show that $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^{1,1}(\mathbb{R}^n) = L^1(\mathbb{R}^n) \cap \dot{F}^{1,2}_1(\mathbb{R}^n)$. But this is well-known, as recalled in Section 5.2.

Proof of (b). It is enough to prove that the space

$$F = \left\{f \in H^{1,1}_{z,0}(\Omega),\ \ d(\mathrm{Supp}\, f, \partial\Omega) > 0\right\}$$

is dense in $H^{1,1}_{z,0}(\Omega)$.

Consider first the case when $\Omega = \mathbb{R}^n_+$. Let $f \in H^{1,1}_{z,0}(\mathbb{R}^n_+)$. The extension of f by 0 outside Ω, which we still denote by f, is in $H^{1,1}(\mathbb{R}^n)$ (see Theorem 25). For all $\varepsilon > 0$, define

$$f_\varepsilon(x) = f(x - \varepsilon e_n).$$

When $\varepsilon \to 0$, $f_\varepsilon \to f$ in $H^{1,1}(\mathbb{R}^n)$, and the f_ε belong to F.

Theorem 24 then yields the case when Ω is special Lipschitz, and an argument using a partition of unity, as for the proof of Theorem 25, gives the general case.

Remark 31. If $E = \left\{\phi \in \mathcal{D}(\mathbb{R}^n);\ \int_{\mathbb{R}^{n-1}} \phi = 0\right\}$, then the space of restrictions to $\mathbb{R}^n_+$ of functions in E is dense in $\dot{H}^{1,1}_z(\mathbb{R}^n_+)$. Indeed, let $f \in \dot{H}^{1,1}_z(\mathbb{R}^n_+)$, $g = \mathrm{Tr}\, f$ and h constructed from $g \in \dot{B}^{0,1}_1(\mathbb{R}^{n-1})$, as in the proof of Proposition 4. Then, $f - h \in \dot{H}^{1,1}_{z,0}(\mathbb{R}^n_+)$, so that we can find a sequence $(\phi_k)_{k\in\mathbb{N}} \in \mathcal{D}(\mathbb{R}^n_+)$ which converges to $f - h$ in $\dot{H}^{1,1}_{z,0}(\mathbb{R}^n_+)$. Since h is given by an explicit normally convergent series, it may be approximated in $\dot{H}^{1,1}_{z,0}(\mathbb{R}^n_+)$ by finite sums, which are elements of $\left\{\phi \in C^1_0(\mathbb{R}^{n-1}) \otimes \mathcal{D}(\mathbb{R});\ \int_{\mathbb{R}^{n-1}} \phi = 0\right\}$. It is enough to regularize by horizontal convolution to conclude.

Note that this is a statement for the homogeneous space $\dot{H}^{1,1}_z(\mathbb{R}^n_+)$. Since it is not clear to us what $\mathrm{Tr}(H^{1,1}_z(\mathbb{R}^n_+))$ is (due to the L^1 norm), we cannot pursue this matter here.

5.4. Dual spaces

In order to describe what the dual spaces of Hardy–Sobolev spaces are, we need to introduce BMO spaces on domains. A locally square-integrable function f on $\mathbb{R}^n$ is said to be in $\mathrm{BMO}(\mathbb{R}^n)$ if

$$\|f\|^2_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q \left|f(x) - f_Q\right|^2 dx < +\infty$$

where the supremum is taken over all the cubes $Q \in \mathbb{R}^n$ with sides parallel to the axes. Here, $f_E = \dfrac{1}{|E|}\displaystyle\int_E f(x)\, dx$ is the mean of f over E.

Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. The space $\mathrm{BMO}_z(\Omega)$ is defined as being the space of all functions in $\mathrm{BMO}(\mathbb{R}^n)$ supported in $\overline{\Omega}$, equipped with the norm $\|f\|_{\mathrm{BMO}_z(\Omega)} = \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

The space $\mathrm{BMO}_r(\Omega)$ is defined as being the space of all restrictions to Ω of functions in $\mathrm{BMO}(\mathbb{R}^n)$. If $f \in \mathrm{BMO}_r(\Omega)$ define $\|f\|_{\mathrm{BMO}_r(\Omega)}$ by

$$\|f\|_{\mathrm{BMO}_r(\Omega)} = \inf \|F\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

the infimum being taken over all the functions $F \in \mathrm{BMO}(\mathbb{R}^n)$ such that $F|_\Omega = f$. It can also be shown that $\mathrm{BMO}_r(\Omega)$ coincides with the BMO space of Coifman–Weiss (see [12]).

The duality theorems for H^1 and BMO spaces are as follows (see [14,8]):

Proposition 32. (a) *The dual of* $H^1(\mathbb{R}^n)$ *is* $\mathrm{BMO}(\mathbb{R}^n)$.

(b) *The dual of* $H^1_r(\Omega)$ *is* $\mathrm{BMO}_z(\Omega)$.

(c) *The dual of* $H^1_z(\Omega)$ *is* $\mathrm{BMO}_r(\Omega)$.

Say that a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $\mathrm{BMO}^{-1}(\mathbb{R}^n)$ if there exist $\phi_0 \in L^\infty(\mathbb{R}^n)$ and $\mathbf{\Phi} \in \mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^n)$ such that $T = \phi_0 - \operatorname{div} \mathbf{\Phi}$. Define

$$\|T\|_{\mathrm{BMO}^{-1}(\mathbb{R}^n)} = \inf \left(\left\| \phi_0 \right\|_\infty + \|\mathbf{\Phi}\|_{\mathrm{BMO}(\mathbb{R}^n)} \right),$$

where the infimum is taken over all functions $\phi_0 \in L^\infty(\mathbb{R}^n)$ and $\mathbf{\Phi} \in \mathrm{BMO}(\mathbb{R}^n, \mathbb{C}^n)$ such that $T = \phi_0 - \operatorname{div} \mathbf{\Phi}$ (note that our definition of $\mathrm{BMO}^{-1}(\mathbb{R}^n)$ is not the standard homogeneous one). We define two negative BMO-Sobolev spaces on Ω.

Say that a distribution $T \in \mathcal{D}'(\Omega)$ belongs to $\mathrm{BMO}_r^{-1}(\Omega)$ if there exist $\phi_0 \in L^\infty(\Omega)$ and $\mathbf{\Phi} \in \mathrm{BMO}_r(\Omega, \mathbb{C}^n)$ such that, for all $f \in \mathcal{D}(\Omega)$,

$$\langle T, f \rangle = \int_\Omega f(x)\phi_0(x)\, dx + \int_\Omega \nabla f(x).\mathbf{\Phi}(x)\, dx. \tag{4}$$

Define

$$\|T\|_{\mathrm{BMO}_r^{-1}(\Omega)} = \inf \left(\left\| \phi_0 \right\|_\infty + \|\mathbf{\Phi}\|_{\mathrm{BMO}_r(\Omega)} \right),$$

the infimum being taken over all functions $\phi_0 \in L^\infty(\Omega)$ and $\mathbf{\Phi} \in \mathrm{BMO}_r(\Omega, \mathbb{C}^n)$ such that (4) holds. One may write $T = \phi_0 - \operatorname{div} \mathbf{\Phi}$ in $\mathcal{D}'(\Omega)$ where div is the divergence operator on Ω.

Say that a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $\mathrm{BMO}_{z,0}^{-1}(\Omega)$ if there exist $\phi_0 \in L^\infty(\Omega)$, $\mathbf{\Phi} \in \mathrm{BMO}_z(\Omega, \mathbb{C}^n)$ and $h \in L^\infty(\partial\Omega, d\sigma)$ such that, for all $f \in \mathcal{D}(\mathbb{R}^n)$,

$$\langle T, f \rangle = \int_\Omega f(x)\phi_0(x)\, dx + \int_\Omega \nabla f(x).\mathbf{\Phi}(x)\, dx + \int_{\partial\Omega} f(x)h(x)\, d\sigma(x). \tag{5}$$

Define

$$\|T\|_{\mathrm{BMO}_{z,0}^{-1}(\Omega)} = \inf \left(\left\| \phi_0 \right\|_\infty + \|\mathbf{\Phi}\|_{\mathrm{BMO}_z(\Omega)} + \|h\|_{L^\infty(d\sigma)} \right),$$

the infimum being taken over all functions $\phi_0 \in L^\infty(\Omega)$, $\mathbf{\Phi} \in \mathrm{BMO}_z(\Omega, \mathbb{C}^n)$ and $h \in L^\infty(\partial\Omega, d\sigma)$ such that (5) holds. One may write $T = \phi_0 - \operatorname{div} \mathbf{\Phi} + h d\sigma$ in $\mathcal{D}'(\mathbb{R}^n)$, where ϕ_0 and $\mathbf{\Phi}$ are extended by 0 outside Ω and div is the divergence operator on $\mathbb{R}^n$.

The duality statements for Hardy–Sobolev spaces are as follows:

Proposition 33. (a) *The dual of* $H_{z,0}^{1,1}(\Omega)$ *is isomorphic to* $\mathrm{BMO}_r^{-1}(\Omega)$. *More precisely, given* $T = \phi_0 - \operatorname{div} \mathbf{\Phi} \in \mathrm{BMO}_r^{-1}(\Omega)$, *the linear functional*

$$\mathcal{D}(\Omega) \ni f \mapsto \int_\Omega f\phi_0 + \int_\Omega \nabla f.\mathbf{\Phi} = \langle T, f \rangle$$

extends by density to a bounded linear functional L_T *on* $H^{1,1}_{z,0}(\Omega)$. *Conversely, for any* $L \in (H^{1,1}_{z,0}(\Omega))'$, *there exists a unique* $T \in \mathrm{BMO}^{-1}_r(\Omega)$ *such that, for all* $f \in \mathcal{D}(\Omega)$, $L(f) = \langle T, f\rangle$. *One has* $\|L_T\| \sim \|T\|_{\mathrm{BMO}^{-1}_r(\Omega)}$.

(b) *The dual of* $H^{1,1}_r(\Omega)$ *is isomorphic to* $\mathrm{BMO}^{-1}_{z,0}(\Omega)$. *More precisely, given* $T = \phi_0 - \operatorname{div}\mathbf{\Phi} + h\,d\sigma \in \mathrm{BMO}^{-1}_{z,0}(\Omega)$, *the linear functional*

$$\mathcal{D}(\mathbb{R}^n) \ni f \mapsto \int_\Omega f\phi_0 + \int_\Omega \nabla f.\mathbf{\Phi} + \int_{\partial\Omega} fh\,d\sigma = \langle T, f\rangle$$

extends by density to a bounded linear functional L_T *on* $H^{1,1}_r(\Omega)$. *Conversely, for all* $L \in (H^{1,1}_r(\Omega))'$, *there exists a unique* $T \in \mathrm{BMO}^{-1}_{z,0}(\Omega)$ *such that, for all* $f \in \mathcal{D}(\mathbb{R}^n)$, $L(f) = \langle T, f\rangle$. *One has* $\|L_T\| \sim \|T\|_{\mathrm{BMO}^{-1}_{z,0}(\Omega)}$.

Let us prove assertion (a). Let $T \in \mathrm{BMO}^{-1}_r(\Omega)$, $\varepsilon > 0$ and $\phi_0 \in L^\infty(\Omega)$, $\mathbf{\Phi} \in \mathrm{BMO}_r(\Omega, \mathbb{C}^n)$ such that (4) holds and $\left\|\phi_0\right\|_\infty + \|\mathbf{\Phi}\|_{\mathrm{BMO}_r(\Omega)} \leqslant (1+\varepsilon)\,\|T\|_{\mathrm{BMO}^{-1}_r(\Omega)}$. Then, for all $f \in \mathcal{D}(\Omega)$,

$$\begin{aligned}|\langle T, f\rangle| &\leqslant \left|\int_\Omega f\phi_0\right| + \left|\int_\Omega \nabla f.\mathbf{\Phi}\right| \\ &\leqslant \left(\left\|\phi_0\right\|_{L^\infty(\Omega)} + \|\mathbf{\Phi}\|_{\mathrm{BMO}_r(\Omega)}\right)\|f\|_{H^{1,1}_{z,0}(\Omega)}.\end{aligned}$$

Since $\mathcal{D}(\Omega)$ is dense in $H^{1,1}_{z,0}(\Omega)$, this means that $f \mapsto \langle T, f\rangle$ extends to a bounded linear form L_T on $H^{1,1}_{z,0}(\Omega)$, with

$$\|L_T\| \leqslant (1+\varepsilon)\,\|T\|_{\mathrm{BMO}^{-1}_r(\Omega)}.$$

Since this is true for all $\varepsilon > 0$, one obtains

$$\|L_T\| \leqslant \|T\|_{\mathrm{BMO}^{-1}_r(\Omega)}.$$

Conversely, let L be a bounded linear form on $H^{1,1}_{z,0}(\Omega)$. Since $H^{1,1}_{z,0}(\Omega)$ is isometrically isomorphic to a subspace of $L^1(\Omega) \oplus H^1_z(\Omega) \oplus \cdots \oplus H^1_z(\Omega)$, there exists $\phi_0 \in L^\infty(\Omega)$ and $\phi_1, \ldots, \phi_n \in \mathrm{BMO}_r(\Omega)$ such that, for all $f \in \mathcal{D}(\Omega)$,

$$L(f) = \int_\Omega f(x)\phi_0(x)\,dx + \sum_{k=1}^n \int_\Omega \partial_k f(x)\phi_k(x)\,dx$$

and

$$\sup_{1\leqslant i\leqslant n}\left(\|\phi_0\|_\infty, \|\phi_i\|_{\mathrm{BMO}_r(\Omega)}\right)\leqslant \|L\|.$$

Set $\mathbf{\Phi}=(\phi_1,\dots,\phi_n)$ and $T=\phi_0-\operatorname{div}\mathbf{\Phi}\in\mathcal{D}'(\Omega)$. Then $T\in \mathrm{BMO}_r^{-1}(\Omega)$, $L=L_T$ and $\|T\|\leqslant(n+1)\,\|L\|$. This proves assertion (a). The proof of assertion (b) is entirely similar.

Remark 34. We also have that the dual of $H^{1,1}_{r,0}(\Omega)$ is isomorphic to

$$\mathrm{BMO}_z^{-1}(\Omega)=\big\{T\in\mathcal{D}'(\mathbb{R}^n);\ \exists\phi_0\in L^\infty(\Omega) \text{ and } \mathbf{\Phi}\in \mathrm{BMO}_z(\Omega,\mathbb{C}^n) \text{ such that } T=\phi_0-\operatorname{div}_{\mathbb{R}^n}(\mathbf{\Phi})\big\}.$$

Since $\mathrm{BMO}_z(\Omega)$ has no trace on $\partial\Omega$, $\operatorname{div}_{\mathbb{R}^n}(\mathbf{\Phi})$ is not identical to $\operatorname{div}_{\Omega}(\mathbf{\Phi})$. For instance, if $\mathbf{e}$ is a constant nonzero vector, $\mathbb{1}_\Omega\mathbf{e}\in\mathrm{BMO}_z(\Omega)$, and

$$\operatorname{div}_{\mathbb{R}^n}(\mathbb{1}_\Omega\mathbf{e})=-\mathbf{e}.\boldsymbol{\nu}\mathbf{d}\boldsymbol{\sigma}\neq\operatorname{div}_\Omega(\mathbb{1}_\Omega\mathbf{e})=0.$$

Remark 35. Because $\mathrm{Tr}(H^{1,1}_r(\Omega))=L^1(\partial\Omega)$ is not a dual space, $H^{1,1}_r(\Omega)$ is not a dual space either.

5.5. Interpolation with classical Sobolev spaces

We easily obtain a characterization of classical Sobolev spaces through our maximal functions:

Theorem 36. *Let* $f\in L^q(\Omega)$, $1<q\leqslant+\infty$.
(a) $f\in W^{1,q}(\Omega)\Leftrightarrow N^{(1)}f\in L^q(\Omega)$, *and* $\|\nabla f\|_q\sim\left\|N^{(1)}f\right\|_q$.
(b) $f\in W^{1,q}_0(\Omega)\Leftrightarrow M^{(1)}f\in L^q(\Omega)$, *and* $\|\nabla f\|_q\sim\left\|M^{(1)}f\right\|_q$.

We first prove (a). If $f\in W^{1,q}(\Omega)$, then $\nabla f\in L^q(\Omega)$. The Green formula yields

$$\int_\Omega f(y)\operatorname{div}\mathbf{\Phi}(y)\,dy=-\langle\nabla f,\mathbf{\Phi}\rangle$$

for all $x\in\Omega$ and all $\mathbf{\Phi}\in G_x(\Omega)$, since $\mathbf{\Phi}.\boldsymbol{\nu}=0$ on $\partial\Omega$. As a consequence, $N^{(1)}f\leqslant C M^{\Omega}_{\mathrm{HL}}(|\nabla f|)$, where M^{Ω}_{HL} was defined in the proof of Proposition 17, hence

$$\left\|N^{(1)}f\right\|_q\leqslant C\,\|\nabla f\|_{L^q(\Omega)}.$$

Conversely, assume that $N^{(1)}f \in L^q(\Omega)$. Then, assertion (a) in Lemma 8 shows that, for all $u \in \mathcal{D}(\Omega)$ and all $1 \leqslant i \leqslant n$,

$$\left| \int_\Omega f(x)\partial_i u(x)\, dx \right| \leqslant C \left\| N^{(1)} f \right\|_{L^q(\Omega)} \|u\|_{L^{q'}(\Omega)},$$

which implies that, for all $1 \leqslant i \leqslant n$, $\partial_i f \in L^q(\Omega)$ and that

$$\left\| \partial_i f \right\|_{L^q(\Omega)} \leqslant C \left\| N^{(1)} f \right\|_{L^q(\Omega)}.$$

The proof of (b) is similar for the direct part. For the converse, assertion (b) in Lemma 8 yields

$$\left| \int_\Omega f(x)\partial_i u(x)\, dx \right| \leqslant C \left\| M^{(1)} f \right\|_{L^q(\Omega)} \|u\|_{L^{q'}(\mathbb{R}^n)},$$

for all $1 \leqslant i \leqslant n$ and all $u \in \mathcal{D}(\mathbb{R}^n)$. As a consequence, $f \in W_0^{1,q}(\Omega)$ and

$$\|\nabla f\|_{L^q(\Omega)} \leqslant C \left\| M^{(1)} f \right\|_{L^q(\Omega)}.$$

Remark 37. Observe that, in (b), the characterization by $M^{(1)}$ encodes the zero trace.

We deduce from this characterization of Sobolev spaces an interpolation theorem between Sobolev and Hardy–Sobolev spaces:

Corollary 38. *Let $1 < q \leqslant \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = (1-\theta) + \frac{\theta}{q}$. For the complex interpolation method, we have*:
(a) $[H_r^{1,1}(\Omega), W^{1,q}(\Omega)]_\theta = W^{1,p}(\Omega)$.
(b) $[H_{z,0}^{1,1}(\Omega), W_0^{1,q}(\Omega)]_\theta = W_0^{1,p}(\Omega)$.

For (a), use the characterization by f and $N^{(1)}f \in L^p$, together with the classical linearization method of maximal operators, found, for instance, in [26, Chapter 5]. For (b), use the characterization by f and $M^{(1)}f \in L^p$.

Remark 39. Theorem 36 and Corollary 38 are valid for $\Omega = \mathbb{R}^n$. In this case, $M^{(1)}$ and $N^{(1)}$ are the same and Corollary 38 is well-known (see [15]), while Theorem 36 is still new.

6. The square root problem: endpoint estimates

Our next application is toward endpoint estimates for the square roots of uniformly elliptic second order differential operators in divergence form.

Let $A \in L^\infty(\mathbb{R}^n, M_n(\mathbb{C}))$. Assume that A is uniformly elliptic, which means that there exists $\delta > 0$ such that, for almost all $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{C}^n$,

$$\operatorname{Re} A(x)\xi.\overline{\xi} \geqslant \delta \left|\xi\right|^2 .$$

If $V(\Omega)$ is a closed subset of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$, there is a unique operator $L = (A, \Omega, V)$ which is the maximal accretive operator associated with the accretive sesquilinear form

$$Q(f, g) = \int_\Omega A(x)\nabla f(x).\overline{\nabla g(x)}\, dx$$

for all $(f, g) \in V \times V$. We are interested in the Dirichlet boundary condition ($V = W_0^{1,2}(\Omega)$) and the Neumann boundary condition ($V = W^{1,2}(\Omega)$). Under Dirichlet (resp. Neumann) boundary condition, L will be denoted by L_{D} (resp. L_{N}).

Such an operator L has a unique maximal-accretive square root $L^{1/2}$ given, for instance, by

$$L^{1/2} f = \frac{2}{\pi}\int_0^{+\infty} (I + t^2 L)^{-1} t L f \frac{dt}{t}, \ \ \forall f \in \mathcal{D}(L), \tag{6}$$

the integral being convergent in $L^2(\Omega)$.

We look for comparison results between the norms of $L^{1/2} f$ and ∇f in suitable Hardy spaces. To that purpose, we have to deal with two assumptions on L, which we now describe.

Say that L satisfies (G) is the following three conditions hold:

The kernel of e^{-tL}, denoted by $K_t(x, y)$, is a measurable function on $\Omega \times \Omega$ and there exist $C_G, \alpha > 0$ such that, for all $0 < t < +\infty$ and almost every $x, y \in \Omega$,

$$|K_t(x, y)| \leqslant \frac{C_G}{t^{n/2}}\, e^{-\alpha \frac{|x-y|^2}{t}} . \tag{7}$$

For all $y \in \Omega$ and all $0 < t < +\infty$, the function $x \mapsto K_t(x, y)$ is Hölder continuous in Ω and there exist $C_H, \mu > 0$ such that, for all $0 < t < +\infty$ and all $x, x', y \in \Omega$,

$$\left|K_t(x, y) - K_t(x', y)\right| \leqslant \frac{C_H}{t^{n/2}} \frac{\left|x - x'\right|^\mu}{t^{\mu/2}}. \tag{8}$$

For all $x \in \Omega$ and all $0 < t < +\infty$, the function $y \mapsto K_t(x, y)$ is Hölder continuous in Ω and there exist $C_H, \mu > 0$ such that, for all $0 < t < +\infty$ and all $y, y', x \in \Omega$,

$$\left|K_t(x, y) - K_t(x, y')\right| \leqslant \frac{C_H}{t^{n/2}} \frac{\left|y - y'\right|^{\mu}}{t^{\mu/2}}. \tag{9}$$

Say that L satisfies (G_{loc}) if (7)–(9) holds for all $0 < t < 1$.

Our endpoint estimates are the following ones:

Theorem 40. *Assume that $L = (A, \Omega, V)$ satisfies (G) if Ω is unbounded or (G_{loc}) if Ω is bounded. Assume furthermore that L satisfies the technical condition (T) when Ω is a general strongly Lipschitz domain (bounded or not). Then:*

(a) *Under Dirichlet boundary condition, there exists $C > 0$ such that, for all $f \in \mathcal{D}(\Omega)$,*

$$C^{-1} \|\nabla f\|_{H^1_z(\Omega)} \leqslant \left\|L_{\mathrm{D}}^{1/2} f\right\|_{H^1_r(\Omega)} \leqslant C\left(\|\nabla f\|_{H^1_z(\Omega)} + \|f\|_{L^1(\Omega)}\right).$$

(b) *Under Neumann boundary condition and if Ω^c is unbounded, there exists $C > 0$ such that, for all $f \in \mathcal{D}(\mathbb{R}^n)\big|_{\Omega}$,*

$$C^{-1} \|\nabla f\|_{H^1_r(\Omega)} \leqslant \left\|L_{\mathrm{N}}^{1/2} f\right\|_{H^1_z(\Omega)} \leqslant C\left(\|\nabla f\|_{H^1_z(\Omega)} + \|f\|_{L^1(\Omega)}\right).$$

Furthermore, if Ω is a special Lipschitz domain, or is bounded, then the inequalities in (a) *and* (b) *hold without the L^1 norms.*

We shall describe the technical condition (T) in Section 6.1.3. At this point, we mention that it is only useful for the right hand inequalities. We also think that this assumption is only due to our method of proof. That Ω^c is unbounded is only used for the left-hand inequality in (b). It is unnecessary for special Lipschitz or bounded domains.

Since $\mathcal{D}(\Omega)$ is dense in $H^{1,1}_{z,0}(\Omega)$ (see Theorem 30), assertion (a) extends to $H^{1,1}_{z,0}$. Similarly, assertion (b) extends to $H^{1,1}_r(\Omega)$.

We make a few comments about Theorem 40.

When $p = 2$, the fact that $V = \mathcal{D}(L^{1/2})$ with the equivalence $\left\|L^{1/2} f\right\|_2 \sim \|\nabla f\|_2$ or its local version $\left\|L^{1/2} f\right\|_2 + \|f\|_2 \sim \|\nabla f\|_2 + \|f\|_2$ (without assumption (G)) has been known for a long time as the Kato conjecture [20]. This problem is now completely solved on $\mathbb{R}^n$ (see [1,2] and the references therein) and on strongly Lipschitz domains (see [5]).

Under assumption (G), it is proved, in [7], that for all $1 < p < +\infty$,

$$\left\|L^{1/2} f\right\|_p \leqslant C\left(\|\nabla f\|_p + \|f\|_p\right) \tag{10}$$

and that there exists $\varepsilon > 0$ depending on ellipticity such that, for all $1 < p < 2 + \varepsilon$,

$$\|\nabla f\|_p \leqslant C \left\| L^{1/2} f \right\|_p . \tag{11}$$

These estimates hold under Dirichlet or Neumann boundary condition, for $f \in V \cap W^{1,p}(\Omega)$. Recall also that, for $1 < p \leqslant 2$, (11) holds under assumption (7) only (see [13]).

Theorem 40 investigates the limit case $p = 1$ in (10) and (11). When $\Omega = \mathbb{R}^n$, it is proved in [4, Chapter 4, Theorem 1], that, if (G) holds for L, $L^{1/2} f$ and ∇f have comparable norms in $H^1(\mathbb{R}^n)$. Theorem 40 of the present paper provides analogous comparisons on a strongly Lipschitz domain Ω. It should be noted that both $H^1_r(\Omega)$ and $H^1_z(\Omega)$ appear in Theorem 40, and that, contrary to the results in [7], the statements depend on the boundary condition. This should be related to [3], in which it is proved that $H^1_r(\Omega)$ (resp. $H^1_z(\Omega)$) is characterized by the nontangential maximal function associated with the Poisson kernel of L subject to the Dirichlet boundary condition (resp. the Neumann boundary condition).

6.1. *Upper estimate for* $\left\| L^{1/2} f \right\|_{H^1}$

In this section, we prove the inequalities $\left\| L^{1/2} f \right\|_{H^1} \leqslant C \left(\|\nabla f\|_{H^1} + \|f\|_1 \right)$ (with the suitable Hardy spaces, depending on the boundary condition) in Theorem 40.

6.1.1. *The case when* $\Omega = \mathbb{R}^n_+$

The Dirichlet case: Let $f \in W^{1,2}_0(\mathbb{R}^n_+)$. Denote by f_o the odd extension of f to $\mathbb{R}^n$ (see Section 2.1 for the notations). Recall that, for all $1 \leqslant i \leqslant n-1$, $\partial_i f_o$ is the odd extension of $\partial_i f$ and, since f has zero trace on $\partial\mathbb{R}^n_+$, $\partial_n f_o$ is the even extension of $\partial_n f$ (in $W^{1,2}(\mathbb{R}^n)$).

Define also

$$\overline{A}(x) = \begin{cases} A(x) & \text{if } x_n > 0, \\ SA(Sx)S & \text{if } x_n < 0, \end{cases}$$

and $\overline{L} = (\overline{A}, \mathbb{R}^n, W^{1,2}(\mathbb{R}^n))$. Then, $\overline{L}(f_o)$ is the odd extension of $L_{\mathrm{D}} f$. Therefore, $(\overline{L})^{1/2}(f_o)$ is the odd extension of $L_{\mathrm{D}}^{1/2} f$. This can be deduced from

$$L_{\mathrm{D}}^{1/2} = c \int_0^{+\infty} L_{\mathrm{D}} e^{-t^2 L_{\mathrm{D}}}\, dt$$

and the analogous formula for $\overline{L}^{1/2}$. Notice that $\overline{L}$ also satisfies (G). The estimate

$$\left\| \overline{L}^{1/2} f_o \right\|_{H^1(\mathbb{R}^n)} \sim \|\nabla f_o\|_{H^1(\mathbb{R}^n)}$$

therefore holds true. Moreover, as recalled in Section 2.1, $H^1_r(\mathbb{R}^n_+)$ is exactly the space of functions defined on $\mathbb{R}^n_+$ whose odd extension belongs to $H^1(\mathbb{R}^n)$. This and Proposition 2 show that

$$\left\| L_{\mathrm{D}}^{1/2} f \right\|_{H^1_r(\mathbb{R}^n_+)} \sim \|\nabla f\|_{H^1_z(\mathbb{R}^n_+)},$$

which completes the proof in this case.

The Neumann case: The proof is entirely similar: one just has to consider the even extension of f instead of its odd extension, and to use the fact that $H^1_z(\mathbb{R}^n_+)$ is exactly the space of functions defined on $\mathbb{R}^n_+$ whose even extension belongs to $H^1(\mathbb{R}^n)$ (see Section 2.1) and Proposition 2.

Remark 41. The same proof as in [4] shows that, if $L = (A, \mathbb{R}^n, W^{1,2}(\mathbb{R}^n))$ satisfies (G_{loc}), and if, for $d > 0$,

$$S = \int_0^d (I + t^2 L)^{-1} t^2 L \, \frac{dt}{t^2},$$

one has

$$\|Sf\|_{H^1(\mathbb{R}^n)} \leqslant C \, \|\nabla f\|_{H^1(\mathbb{R}^n)}.$$

It follows that, when $\Omega = \mathbb{R}^n_+$, if $L = (A, \Omega, V)$ satisfies (G_{loc}), and, for $d > 0$,

$$S = \int_0^d (I + t^2 L)^{-1} t^2 L \, \frac{dt}{t^2},$$

one has

$$\|Sf\|_{H^1_*(\Omega)} \leqslant C \, \|\nabla f\|_{H^1_\sharp(\Omega)},$$

where $(*, \sharp) = (r, z)$ under DBC and $(*, \sharp) = (z, r)$ under NBC.

6.1.2. The case when Ω is special Lipschitz

Consider now a special Lipschitz domain Ω, and the mapping h of Section 5.1. Set $V(\Omega) = W^{1,2}_0(\Omega)$ under Dirichlet boundary condition, and $V(\Omega) = W^{1,2}(\Omega)$ under Neumann boundary condition. If $f \in V(\Omega)$, Theorem 24 shows that

$$\|\nabla f\|_{H^1(\Omega)} \sim \|\nabla (f \circ h)\|_{H^1(\mathbb{R}^n_+)}.$$

where H^1 denotes either H^1_r or H^1_z in both sides of the inequality. Moreover, if $A_h(x) = (J_h(x)^{-1})^t A(h(x)) J_h(x)^{-1}$ for all $x \in \mathbb{R}^n_+$ and $L_h = (A_h, \mathbb{R}^n_+, V(\mathbb{R}^n_+))$, since $|\det J_h(x)| = 1$, one has, for all $f \in \mathcal{D}(L)$, $f \circ h \in \mathcal{D}(L_h)$ and

$$L_h(f \circ h) = Lf,$$

which implies that, for all $f \in V(\Omega)$, $f \circ h \in V(\mathbb{R}^n_+)$ and

$$L_h^{1/2}(f \circ h) = L^{1/2} f.$$

Finally, observing that L_h satisfies (G) and using the result proved in the case when $\Omega = \mathbb{R}^n_+$, one obtains

$$\left\| L_{\mathrm{D}}^{1/2} f \right\|_{H^1_r(\Omega)} \sim \|\nabla f\|_{H^1_z(\Omega)},$$

and

$$\left\| L_{\mathrm{N}}^{1/2} f \right\|_{H^1_z(\Omega)} \sim \|\nabla f\|_{H^1_r(\Omega)}.$$

Before going further, notice that, when Ω is special Lipschitz, we have proved a comparison result for the Sobolev homogeneous seminorms:

Theorem 42. *Let Ω be a special Lipschitz domain. Assume that $L = (A, \Omega, V)$ satisfies (G). Then*:

(a) *Under Dirichlet boundary condition, for all $f \in \mathcal{D}(\Omega)$,*

$$\|\nabla f\|_{H^1_z(\Omega)} \sim \left\| L^{1/2} f \right\|_{H^1_r(\Omega)}.$$

(b) *Under Neumann boundary condition, for all $f \in \mathcal{D}(\mathbb{R}^n)$,*

$$\|\nabla f\|_{H^1_r(\Omega)} \sim \left\| L^{1/2} f \right\|_{H^1_z(\Omega)}.$$

Remark 43. Using the final remark in the previous section, one proves similarly that, when Ω is special Lipschitz, if $L = (A, \Omega, V)$ satisfies (G_{loc}) and, for $d > 0$,

$$S = \int_0^d (I + t^2 L)^{-1} t^2 L \, \frac{dt}{t^2},$$

one has

$$\|Sf\|_{H^1_*(\Omega)} \leqslant C\, \|\nabla f\|_{H^1_\sharp(\Omega)}\,,$$

where $(*, \sharp) = (r, z)$ under DBC and $(*, \sharp) = (z, r)$ under NBC.

6.1.3. General strongly Lipschitz domains

We begin with two lemmata.

Lemma 44. *If $f \in W^{1,1}(\Omega)$, then $f \in h^1_z(\Omega)$.*

Proof. Extend f by 0 outside Ω. We show that $f \in h^1(\mathbb{R}^n)$. By the Sobolev embedding, one has

$$\sum_k \left\| f\chi_k \right\|_{L^p(\mathbb{R}^n)} \leqslant C \left(\|f\|_{L^1(\Omega)} + \|\nabla f\|_{L^1(\Omega)} \right),$$

where $p = \frac{n}{n-1}$ and χ_k is a C^∞ partition of unity associated to a covering of $\mathbb{R}^n$ by balls with radii 1 and the support of each χ_k is contained in a ball with radius 2. Since

$$\left\| f\chi_k \right\|_{h^1(\mathbb{R}^n)} \leqslant C \left\| f\chi_k \right\|_{L^p(\mathbb{R}^n)} \left(1 + \left|\operatorname{supp} \chi_k\right|^{1/p'} \right),$$

it follows that $f \in h^1(\mathbb{R}^n)$ and $\|f\|_{h^1(\mathbb{R}^n)} \leqslant C \left(\|f\|_{L^1(\Omega)} + \|\nabla f\|_{L^1(\Omega)} \right)$. $\square$

Lemma 45. *For all $t \geqslant 1$ and $y \in \mathbb{R}^n$, the function $\Omega \ni x \mapsto \frac{1}{t^n} e^{-\alpha|x-y|/t}$ belongs to $h^1_z(\Omega)$ and its norm only depends on $\alpha > 0$.*

Proof. Let g be the extension by 0 outside Ω of this function. Consider χ_k as in the proof of Lemma 44 and let x_k be the center of the ball with radius 1 associated with χ_k. For all $x \in \operatorname{Supp} \chi_k \subset B(x_k, 2)$,

$$\frac{1}{t^n} e^{-\alpha|x-y|/t} \leqslant \frac{e^{2\alpha}}{t^n}\, e^{-\alpha|x_k-y|/t},$$

since

$$\frac{|x-y|}{t} \geqslant \frac{|x_k - y|}{t} - \frac{|x - x_k|}{t} \geqslant \frac{|x_k - y|}{t} - 2.$$

It follows that, for all k,

$$\left\| g\chi_k \right\|_{h^1(\mathbb{R}^n)} \leqslant C\, \frac{e^{2\alpha}}{t^n}\, e^{-\alpha|x_k-y|/t} (1 + 2^n).$$

Therefore,

$$\|g\|_{h^1(\mathbb{R}^n)} \leqslant C \sum_k \frac{1}{t^n} e^{-\alpha|x_k-y|/t} \leqslant C'. \quad \square$$

Corollary 46. *If L satisfies (G) (more precisely, if L satisfies the upper Gaussian estimate (7)), for all $d>0$, one has*

$$\left\| \int_d^{+\infty} (I+t^2L)^{-1}t^2Lf \, \frac{dt}{t^2} \right\|_{h^1_z(\Omega)} \leqslant C \left(\|f\|_{L^1(\Omega)} + \|\nabla f\|_{L^1(\Omega)} \right).$$

Proof. Since L satisfies (G), the kernel of $(I+t^2L)^{-1}$ is controlled by $\frac{C}{t^n}e^{-\alpha|x-y|/t}$. Write $(I+t^2L)^{-1}t^2L = I-(I+t^2L)^{-1}$, and use Lemmas 44 and 45. $\square$

This result shows at once that, for $\Omega=\mathbb{R}^n$ or Ω strongly Lipschitz, if L satisfies (G) (in fact, (7) is sufficient), then, for all $d>0$,

$$\left\| \int_d^{+\infty} (I+t^2L)^{-1}t^2Lf \, \frac{dt}{t^2} \right\|_{h^1_*(\Omega)} \leqslant C \left(\|f\|_{L^1(\Omega)} + \|\nabla f\|_{H^1_\sharp(\Omega)} \right),$$

where $(*,\sharp)=(r,z)$ under DBC and $(*,\sharp)=(z,r)$ under NBC.

We assume now that Ω is an unbounded strongly Lipschitz domain, that L satisfies (G) and the following technical condition (T): there exist Ω_k, χ_k, η_k, $0\leqslant k\leqslant s$, and $d>0$, as in Section 5.2, such that $L_k=(A,\Omega_k,V_k)$ satisfy the Gaussian upper estimate for all $t<d$, where $V_k=W^{1,2}(\Omega_k)$ if $V=W^{1,2}(\Omega)$ and $V_k=W^{1,2}_0(\Omega_k)$ if $V=W^{1,2}_0(\Omega)$. This condition is satisfied in all cases where we know that (G) holds (see [6]).

Write

$$L^{1/2}f = c\int_0^d (I+t^2L)^{-1}t^2Lf\frac{dt}{t^2} + c\int_d^{+\infty}(I+t^2L)^{-1}t^2Lf\frac{dt}{t^2},$$

and, for all $t<d$,

$$\begin{aligned}(I+t^2L)^{-1}t^2Lf(x) &= \sum_k \eta_k(x)v_t^k(x) + \sum_k (1-\eta_k(x))(I+t^2L)^{-1}t^2Lf_k(x)\\ &\quad + \sum_k \eta_k(x)((I+t^2L)^{-1}t^2L(f_k)(x)-v_t^k(x))\\ &= A+B+C,\end{aligned}$$

where $f_k = f\chi_k$,

$$\widetilde{f_k}(x) = \begin{cases} f_k(x) & \text{if } x \in \Omega \cap \Omega_k, \\ 0 & \text{if } x \in \Omega_k \backslash \Omega \end{cases}$$

and

$$v_t^k(x) = \begin{cases} (I + t^2 L_k)^{-1} t^2 L_k \widetilde{f_k}(x) & \text{if } x \in \Omega \cap \Omega_k, \\ 0 & \text{if } x \in \Omega \backslash \Omega_k. \end{cases}$$

Define also

$$S = \int_0^d (I + t^2 L)^{-1} t^2 L \frac{dt}{t^2}, \quad S_k = \int_0^d (I + t^2 L_k)^{-1} t^2 L_k \frac{dt}{t^2}.$$

Term A: Assume that $k \geqslant 1$. We have seen in the proof of Theorem 25 that $\nabla \widetilde{f_k} \in H^1_*(\Omega_k)$ and that

$$\| \nabla \widetilde{f_k} \|_{H^1_*(\Omega_k)} \leqslant C \left(\| \nabla f \|_{H^1_*(\Omega)} + \| f \|_{L^1(\Omega)} \right),$$

where $* = r$ under DBC. The argument is even simpler under NBC with $* = z$ (and left to the reader). Since (T) holds, the results already obtained in the special Lipschitz case yield

$$\| S_k \widetilde{f_k} \|_{H^1_*(\Omega_k)} \leqslant C \left(\| \nabla f \|_{H^1_\sharp(\Omega)} + \| f \|_{L^1(\Omega)} \right),$$

where $(*, \sharp) = (r, z)$ under DBC, $(*, \sharp) = (z, r)$ under NBC. As a consequence, since $\operatorname{Supp} \eta_k \cap \Omega_k \subset \Omega$, one has

$$\left\| \eta_k \int_0^d v_t^k \frac{dt}{t^2} \right\|_{h^1_*(\Omega)} \leqslant C \left(\| \nabla f \|_{H^1_\sharp(\Omega)} + \| f \|_{L^1(\Omega)} \right).$$

For $k = 0$, we obtain the same thing, using the result on $\mathbb{R}^n$.

Term B: In the analysis of this term and of term C, we only need the fact that $\nabla f \in L^1(\Omega)$. As in [7, p. 613], write

$$(1 - \eta_k) S(\chi_k f) = \int_0^d g_t^k \frac{dt}{t^2},$$

where $g_t^k = (1 - \eta_k)(1 + t^2 L)^{-1}(\chi_k f)$.

Let Q be the unit cube in $\mathbb{R}^n$ and, for all $j \in \mathbb{Z}^n$, let $Q_j = Q + j$. Fix a smooth partition of unity $(\phi_j)_{j\in\mathbb{Z}^n}$ associated with the Q_j's. Set

$$\psi_{k,j} = (1-\eta_k)\phi_j, \;\; \chi_{k,j} = \chi_k\phi_j.$$

The L^1-boundedness of the resolvent of L yields, for all $j, j' \in \mathbb{Z}^n$,

$$\left\| \psi_{k,j}(1+t^2L)^{-1}(\chi_{k,j'}f) \right\|_{L^1(\Omega)} \leqslant C \left\| \chi_{k,j'}f \right\|_{L^1(\Omega)}.$$

Moreover, the off-diagonal estimates given in the proof of Lemma 5 in [5] show that, for all integer $m \geqslant 0$ and all $j, j' \in \mathbb{Z}^n$,

$$\left\| \psi_{k,j}(1+t^2L)^{-1}(\chi_{k,j'}f) \right\|_{L^2(\Omega)} \leqslant C_m \frac{t^m}{(d+|j-j'|)^m} \left\| \chi_{k,j'}f \right\|_{L^2(\Omega)}.$$

Indeed, the distance from the support of $\psi_{k,j}$ and the support of $\chi_{k,j'}$ is at least $c\left(d+\left|j-j'\right|\right)$. By interpolation and integration with respect to t on $[0,d]$, one obtains

$$\left\| \psi_{k,j}S(\chi_{k,j'}f) \right\|_{L^p(\Omega)} \leqslant \frac{C}{(d+|j-j'|)^m} \left\| \chi_{k,j'}f \right\|_{L^p(\Omega)}$$

for $p = \frac{n}{n-1}$ and all $m \geqslant 0$. Since $\psi_{k,j}L^{1/2}(\chi_{k,j'}f)$ is supported in Q_j, this estimate implies that

$$\left\| (1-\eta_k)S(\chi_{k,j'}f) \right\|_{h^1_z(\Omega)} \leqslant \frac{C}{d^m} \left\| \chi_{k,j'}f \right\|_{L^p(\Omega)}.$$

Summing over $j' \in \mathbb{Z}^n$ yields

$$\left\| (1-\eta_k)S(\chi_k f) \right\|_{h^1_z(\Omega)} \leqslant C \left\| f \right\|_{L^p(\Omega)} \leqslant C \left(\left\| \nabla f \right\|_{L^1(\Omega)} + \left\| f \right\|_{L^1(\Omega)} \right).$$

Term C: The treatment is similar. Following [7, p. 612], define

$$u_t^k = (1+t^2L)^{-1}t^2Lf_k, \;\; w_t^k = \eta_k(u_t^k - v_t^k),$$

with v_t^k as above. Consider functions ϕ_j as before, and, for all $j \in \mathbb{Z}$, set

$$v_t^{k,j}(x) = \begin{cases} (1+t^2L_k)^{-1}t^2L_k(\phi_j\widetilde{f_k})(x) & \text{if } x \in \Omega \cap \Omega_k, \\ 0 & \text{if } x \in \Omega \backslash \Omega_k \end{cases}$$

and

$$u_t^{k,j} = (1+t^2L)^{-1}t^2L(\phi_j f_k), \quad w_t^{k,j} = \eta_k(u_t^{k,j} - v_t^{k,j}).$$

Since, on $\Omega \cap \Omega_k$, $u_t^{k,j} = \phi_j f_k - (1+t^2L)^{-1}(\phi_j f_k)$, $v_t^{k,j} = \phi_j \widetilde{f}_k - (1+t^2L_k)^{-1}(\phi_j \widetilde{f}_k)$ and $f_k = \widetilde{f}_k$, one has

$$w_t^{k,j} = \eta_k \left((1+t^2L_k)^{-1}(\phi_j \widetilde{f}_k) - (1+t^2L)^{-1}(\phi_j f_k) \right) \text{ on } \Omega.$$

Observe moreover that $\operatorname{Supp} \eta_k \cap \Omega \subset \Omega \cap \Omega_k$ and $\operatorname{Supp} \widetilde{f}_k \subset \Omega$. The L^1-boundedness of the resolvents of L and L_k therefore yields

$$\left\| w_t^{k,j'} \right\|_{L^1(\Omega)} \leqslant C \left\| \phi_{j'} f_k \right\|_{L^1(\Omega)}.$$

When $\operatorname{dist}(\operatorname{Supp} \phi_j, \ \operatorname{Supp} \phi_{j'}) \leqslant 2$, Lemma 2 in [5] shows that, for all $m \in \mathbb{N}$,

$$\left\| \phi_j w_t^{k,j'} \right\|_{L^2(\Omega)} \leqslant C_m \frac{t^m}{d^m} \left\| \phi_{j'} f_k \right\|_{L^2(\Omega)}.$$

When $\operatorname{dist}(\operatorname{Supp} \phi_j, \ \operatorname{Supp} \phi_{j'}) > 2$, the estimates given in the proof of Lemma 5 in [5] separately for $u_t^{k,j'}$ and $v_t^{k,j'}$ imply that, for all $m \in \mathbb{N}$,

$$\left\| \phi_j w_t^{k,j'} \right\|_{L^2(\Omega)} \leqslant C_m \frac{t^m}{|j-j'|^m} \left\| \phi_{j'} f_k \right\|_{L^2(\Omega)}.$$

One concludes as for term B. $\square$

Up to now, we have proved that

$$\left\| L^{1/2} f \right\|_{h^1_*(\Omega)} \leqslant C \left(\| \nabla f \|_{H^1_\sharp(\Omega)} + \| f \|_{L^1(\Omega)} \right),$$

with $(*, \sharp) = (r, z)$ under DBC and $(*, \sharp) = (z, r)$ under NBC. To conclude, we rely on the following result:

Lemma 47. *Let $f \in L^1(\Omega)$ such that $L^{1/2} f \in h^1_*(\Omega)$, where $* = r$ or $* = z$. Then, $L^{1/2} f \in H^1_*(\Omega)$ and*

$$\left\| L^{1/2} f \right\|_{H^1_*(\Omega)} \leqslant C \left(\left\| L^{1/2} f \right\|_{h^1_*(\Omega)} + \| f \|_{L^1(\Omega)} \right).$$

Indeed, set $g = L^{1/2} f$ and, for all $x \in \Omega$, define

$$g^*(x) = \sup_{(y,t)\in\Omega\times]0,+\infty[,\ \ |y-x|<t} |P_t g(y)|$$

and

$$g^*_{\mathrm{loc}}(x) = \sup_{(y,t)\in\Omega\times]0,1[,\ \ |y-x|<t} |P_t g(y)|$$

where $P_t = e^{-tL^{1/2}}$, $L = L_{\mathrm{D}}$ when $L^{1/2} f \in h^1_r(\Omega)$ and $L = L_{\mathrm{N}}$ when $L^{1/2} f \in h^1_z(\Omega)$. Then, according to [3], $g \in H^1_*(\Omega)$ if and only if $g^* \in L^1(\Omega)$ and $g \in h^1_*(\Omega)$ if and only if $g^*_{\mathrm{loc}} \in L^1(\Omega)$ with equivalence of norms. Thus, it is enough to prove that

$$\|g^*\|_{L^1(\Omega)} \leqslant C \left(\|g^*_{\mathrm{loc}}\|_{L^1(\Omega)} + \|g\|_{L^1(\Omega)} \right).$$

To this end, if $x, y \in \Omega$ with $|y - x| < t$ and $t > 1$, one has, using the upper estimate for the kernel of P_t (see [3, Appendix A]),

$$\begin{aligned}
\left| P_t L^{1/2} f(y) \right| &\leqslant \int_\Omega \frac{|f(z)|}{(t + |y-z|)^{n+1}}\, dz \\
&\leqslant \int_\Omega \frac{|f(z)|}{(t + |x-z|)^{n+1}}\, dz \\
&\leqslant \int_\Omega \frac{|f(z)|}{(1 + |x-z|)^{n+1}}\, dz \\
&= h(x)
\end{aligned}$$

and, by Fubini, one has

$$\|h\|_1 \leqslant C \, \|f\|_1 .$$

Since $g^* \leqslant g^*_{\mathrm{loc}} + h$, Lemma 47 is proved.

We consider now the case when Ω is bounded and L satisfies (G_{loc}). We rely on the following lemma:

Lemma 48. *For all* $t \geqslant 1$ *and* $y \in \mathbb{R}^n$, *the function* $\Omega \ni x \mapsto e^{-\alpha|x-y|/t}$ *belongs to* $h^1_z(\Omega)$ *and its norm only depends on* $|\Omega|$.

Indeed, up to a multiplicative constant only depending on $|\Omega|$, its zero extension is an $h^1(\mathbb{R}^n)$-atom, supported in a cube Q containing Ω with sidelength greater than 1.

We deduce from Lemmas 44 and 48 that, if L satisfies (G_{loc}),

$$\left\| \int_d^{+\infty} (I+t^2L)^{-1} t^2 Lf \, \frac{dt}{t^2} \right\|_{h^1_*(\Omega)} \leqslant C \left(\|f\|_{L^1(\Omega)} + \|\nabla f\|_{H^1_\sharp(\Omega)} \right),$$

where $(*, \sharp) = (r, z)$ under DBC and $(*, \sharp) = (z, r)$ under NBC. Indeed, since (G_{loc}) holds, one has

$$|K_t(x,y)| \leqslant C e^{-c|x-y|^2/t}$$

for all $t > d$ (see [3, Lemma 24]), and it follows that the kernel of $(I+t^2L)^{-1}$ is controlled by $e^{-\alpha|x-y|/t}$.

The treatment of $\int_0^d (I+t^2L)^{-1} t^2 Lf \, \frac{dt}{t^2}$ is similar to the case when Ω is unbounded and (G) holds. Thus, using Lemma 47, one obtains

$$\left\| L^{1/2} f \right\|_{H^1_*(\Omega)} \leqslant C \left(\|\nabla f\|_{H^1_\sharp(\Omega)} + \|f\|_{L^1(\Omega)} \right),$$

where $(*, \sharp) = (r, z)$ under DBC and $(*, \sharp) = (z, r)$ under NBC. Finally, since Ω is bounded, the L^1-norm of f can be dropped. Indeed, for the Dirichlet case, if $f \in \mathcal{D}(\Omega)$, the Poincaré inequality yields $\|f\|_{L^1(\Omega)} \leqslant C \|\nabla f\|_{L^1(\Omega)} \leqslant C \|\nabla f\|_{H^1_z(\Omega)}$. For the Neumann case, if $f \in \mathcal{D}(\mathbb{R}^n)$, since $L_{\mathrm{N}}^{1/2}$ annihilates constants, one may assume that f has zero integral on Ω, and the Poincaré inequality shows again that $\|f\|_{L^1(\Omega)} \leqslant C \|\nabla f\|_{L^1(\Omega)} \leqslant C \|\nabla f\|_{H^1_r(\Omega)}$.

6.2. Riesz transforms

In the present section, we prove the inequalities of the form $\|\nabla f\|_{H^1_*(\Omega)} \leqslant C \left\| L^{1/2} f \right\|_{H^1_\sharp(\Omega)}$ in Theorem 40. Observe that, in the case when Ω is special Lipschitz, we have already obtained equivalences of the form $\|\nabla f\|_{H^1_*(\Omega)} \sim \left\| L^{1/2} f \right\|_{H^1_\sharp(\Omega)}$. However, the arguments in the present section work for a general strongly Lipschitz domain Ω.

Consider the operator $T = \nabla L^{-1/2}$. We make use of the following integral representation:

$$Tf = c \int_0^{+\infty} \nabla e^{-tL} f \, \frac{dt}{\sqrt{t}}$$

valid for all $f \in \mathcal{D}(L)$. We will also make use of the truncated operators T_ε:

$$T_\varepsilon f = c \int_\varepsilon^{1/\varepsilon} \nabla e^{-tL} f \, \frac{dt}{\sqrt{t}}$$

for all $0 < \varepsilon < 1$, which converge strongly to T in $L^2(\Omega)$ when ε goes to zero.

6.2.1. The Dirichlet case

It is enough to show the following result (see the proof of Lemma 11 for the definitions of atoms):

Lemma 49. *If a is an interior or a boundary atom in $H^1_r(\Omega)$, $\|Ta\|_{H^1_z(\Omega)} \leqslant C$.*

Let a be an interior atom, supported in a cube Q. We first prove that, for all $\varepsilon > 0$, $T_\varepsilon a \in L^1$. It is well-known that this is true whenever T_ε is L^2-bounded and when its kernel k_ε satisfies the integral Hörmander condition: there exists $C > 0$ such that, for all $y, y_0 \in \Omega$,

$$\int_{x\in\Omega,\ |x-y|\geqslant 2|y_0-y|} |k_\varepsilon(x,y) - k_\varepsilon(x,y_0)|\ dx \leqslant C.$$

It is easy to see that this condition actually holds when the kernel of e^{-tL} satisfies (7) and (9). Indeed,

$$k_\varepsilon(x,y) = \int_\varepsilon^{1/\varepsilon} \nabla_x K_t(x,y)\,\frac{dt}{\sqrt{t}}$$

and one uses Proposition A.1 in Appendix A.

Fix $\varepsilon > 0$. One has $\int_\Omega T_\varepsilon a(x)\,dx = 0$. Indeed, since a has mean-value zero, one has

$$T_\varepsilon a(x) = \int_\varepsilon^{1/\varepsilon} \int_Q a(y)\nabla_x \left(K_t(x,y) - K_t(x,y_Q)\right)\,dy\,\frac{dt}{\sqrt{t}},$$

where y_Q is the center of Q. One uses the Fubini theorem and an integration by parts to obtain, for any $t \in]\varepsilon, 1/\varepsilon[$,

$$\begin{aligned}\int_\Omega \nabla_x \left(K_t(x,y) - K_t(x,y_Q)\right)\,dx &= \int_{\partial\Omega} \left(K_t(x,y) - K_t(x,y_Q)\right)\mathbf{v}(x)\,d\sigma(x)\\ &= 0.\end{aligned}$$

Next, by computations analogous to the ones in [4, Chapter 4, Lemma 11], whenever ϕ is continuous and has compact support in Ω, one has

$$\begin{aligned}\left|\int_\Omega T_\varepsilon a(x)\phi(x)\,dx\right| &= \left|\int_\Omega T_\varepsilon a(x)(\phi(x) - \phi_Q)\,dx\right|\\ &\leqslant C\,\|\phi\|_{\mathrm{BMO}_r(\Omega)},\end{aligned}$$

where the constant C does not depend on ε. Letting ε go to zero, one obtains

$$\left|\int_{\Omega} Ta(x)\phi(x)\,dx\right| \leqslant C\,\|\phi\|_{\mathrm{BMO}_r(\Omega)},$$

which proves that $\|Ta\|_{H^1_z(\Omega)} \leqslant C$ by duality arguments (see [3, Section 3.4] or [8]).

We now turn to the case of a boundary atom a, which means that a is supported in a cube Q such that $2Q \subset \Omega$ but $4Q \cap \partial\Omega \neq \emptyset$ (recall that a is not assumed to have mean value zero). We first prove as before that $T_\varepsilon a \in L^1(\Omega)$, using the following representation:

$$T_\varepsilon a(x) = \int_\varepsilon^{1/\varepsilon} \int_Q \left(\nabla_x K_t(x,y) - \nabla_x K_t(x,\overline{y_Q})\right) a(y)\,dy,$$

where y_Q is the center of Q and $\overline{y_Q}$ is a point on $\partial\Omega$ such that $\left|y_Q - \overline{y_Q}\right| \sim l_Q$. Next, one concludes, as in the previous case, that $\int_\Omega T_\varepsilon a(x)\,dx = 0$ and that $\|Ta\|_{H^1_z(\Omega)} \leqslant C$.

6.2.2. *The Neumann case*

We first consider the case when Ω is unbounded and (G) holds for all $t > 0$. Consider an interior atom a, supported in a cube Q. Denote by l the sidelength of Q and by δ the distance from the center of Q to $\partial\Omega$. As in the Dirichlet case, one sees that, for all $\varepsilon > 0$, $T_\varepsilon a \in L^1(\Omega)$.

Consider then any function ϕ continuous and compactly supported in Ω, and assume that $\|\phi\|_{\mathrm{BMO}_z(\Omega)} \leqslant 1$. One has

$$\int_\Omega T_\varepsilon a(x)\phi(x)\,dx = \int_\Omega T_\varepsilon a(x)\left(\phi(x) - \phi_Q\right)\,dx + \phi_Q \int_\Omega T_\varepsilon a(x)\,dx.$$

The first term is estimated as in the Dirichlet case. As for the second term, since $\left|\phi_Q\right| \leqslant C\log(\frac{\delta}{l})$, (see [3, Lemma 15], this is where we use that Ω^c is unbounded), one just has to prove uniformly in ε that

$$\left|\int_\Omega T_\varepsilon a(x)\,dx\right| \leqslant \frac{C}{\log(\frac{\delta}{l})}. \tag{12}$$

One has

$$T_\varepsilon a(x) = \int_\varepsilon^{1/\varepsilon} \int_Q a(y)\nabla_x\left(K_t(x,y) - K_t(x,y_Q)\right)\,dy\,\frac{dt}{\sqrt{t}}$$

and we are going to estimate, for all fixed $t > 0$,

$$\left|\int_{\Omega} \nabla_x \left(K_t(x, y) - K_t(x, y_Q)\right) dx\right|.$$

An integration by parts shows that

$$\left|\int_{\Omega} \nabla_x \left(K_t(x, y) - K_t(x, y_Q)\right) dx\right| = \left|\int_{\partial\Omega} \left(K_t(x, y) - K_t(x, y_Q)\right) \vec{n}(x)\, d\sigma(x)\right|.$$

Using (9), one obtains, for some $\mu > 0$,

$$\begin{aligned}
&\left|\int_{\partial\Omega} \left(K_t(x, y) - K_t(x, y_Q)\right) \mathbf{v}(x)\, d\sigma(x)\right| \\
&\leqslant \int_{\partial\Omega} \left|K_t(x, y) - K_t(x, y_Q)\right| \, d\sigma(x) \\
&\leqslant \frac{C}{t^{n/2}} \left(\frac{l}{\sqrt{t}}\right)^{\mu} \int_{\partial\Omega} e^{-c\frac{|x-y_Q|^2}{t}}\, d\sigma(x) \\
&\leqslant \frac{C}{t^{n/2}} \left(\frac{l}{\sqrt{t}}\right)^{\mu} \sum_{k=0}^{+\infty} \int_{x\in\partial\Omega, 2^k\delta \leqslant |x-y_Q| < 2^{k+1}\delta} e^{-c\frac{|x-y_Q|^2}{t}}\, d\sigma(x) \\
&\leqslant \frac{C}{t^{1/2}} \left(\frac{l}{\sqrt{t}}\right)^{\mu} \sum_{k=0}^{+\infty} e^{-c\frac{2^{2k}\delta^2}{t}} \left(\frac{2^k\delta}{\sqrt{t}}\right)^{n-1}.
\end{aligned}$$

As a consequence, using Fubini's theorem,

$$\begin{aligned}
\left|\int_{\Omega} T_\varepsilon a(x)\, dx\right| &\leqslant C \sum_{k=0}^{+\infty} \int_0^{+\infty} \left(\frac{l}{\sqrt{t}}\right)^{\mu} e^{-c\frac{2^{2k}\delta^2}{t}} \left(\frac{2^k\delta}{\sqrt{t}}\right)^{n-1} \frac{dt}{t} \\
&= C \left(\frac{l}{\delta}\right)^{\mu} \left(\int_0^{+\infty} u^{-\frac{\mu+n+1}{2}} e^{-\frac{c}{u}}\, du\right) \sum_{k=0}^{+\infty} 2^{-2k\mu} \\
&\leqslant C \left(\frac{l}{\delta}\right)^{\mu}
\end{aligned}$$

and this estimate is uniform in $\varepsilon > 0$. Finally, letting ε go to zero, we obtain

$$\left| \int_\Omega Ta(x)\phi(x)\,dx \right| \leqslant C,$$

and this yields $\|Ta\|_{H^1_r(\Omega)} \leqslant C$ by a duality argument (see [3, Section 3.4]).

Assume now that Ω is bounded and that the estimates for K_t only hold for small time. Following [7], we write

$$T = \int_0^1 \nabla e^{-tL} \frac{dt}{\sqrt{t}} + \int_1^{+\infty} \nabla e^{-tL} \frac{dt}{\sqrt{t}} = T_1 + T_2.$$

The treatment of the first integral is as before, and yields

$$\|T_1 f\|_{H^1_r(\Omega)} \leqslant C \|f\|_{H^1_z(\Omega)}.$$

Moreover, one has

$$\|T_2 f\|_{L^2(\Omega)} \leqslant C \|f\|_{L^1(\Omega)}. \tag{13}$$

Indeed,

$$T_2 f = \int_1^{+\infty} \nabla e^{-\frac{t}{2}L} e^{-\frac{t}{2}L} f \frac{dt}{\sqrt{t}}.$$

Since $\left\|e^{-sL} f\right\|_2 \leqslant Cs^{-n/4} \|f\|_1$ and $\left\|\nabla e^{-sL} g\right\|_2 \leqslant Cs^{-1/2} \|g\|_2$, we obtain (13). Thus, $T_2 f \in h^1_r(\Omega)$ (recall that Ω is bounded), and since $h^1_r(\Omega) = H^1_r(\Omega)$ (see [3, Remark 17]), we have

$$\|T_2 f\|_{H^1_r(\Omega)} \leqslant C \|f\|_{L^1(\Omega)} \leqslant C \|f\|_{H^1_z(\Omega)}.$$

The proof is complete.

Appendix A. Kernel estimates

We summarize some estimates about $\nabla K_t(x, y)$ which follow from assumptions (7)–(9) (see [7, Proposition 15]):

Proposition A.1. (a) *For all* $x, x' \in \Omega$, *all* $0 < t < \tau$ *and all* $r > 0$ *with* $|x - x'| \leqslant r/2$,

$$\left(\int_{r \leqslant |x-y| \leqslant 2r} \left|\nabla_y K_t(x, y) - \nabla_y K_t(x', y)\right|^2 dy\right)^{1/2}$$

$$\leqslant c\,(C_G + C_H)\, t^{-\frac{1}{2}-\frac{n}{4}} \left(\frac{|x - x'|}{\sqrt{t}}\right)^{\eta} \left(\frac{r}{\sqrt{t}}\right)^{\frac{n-2}{2}} e^{-\beta\frac{r^2}{t}}.$$

(b) *For all* $x, x' \in \Omega$, *all* $z \in \mathbb{R}^n$, *all* $0 < t < \tau$ *and all* $r > 0$,

$$\left(\int_{|z-y| \leqslant r} \left|\nabla_y K_t(x, y) - \nabla_y K_t(x', y)\right|^2 dy\right)^{1/2}$$

$$\leqslant c C_H t^{-\frac{1}{2}-\frac{n}{4}} \left(\frac{|x - x'|}{\sqrt{t}}\right)^{\eta} \left(\frac{r}{\sqrt{t}}\right)^{\frac{n-2}{2}}.$$

In this proposition, all the integrals are computed on Ω.

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